

# FULLY REPRESENTABLE AND \*-SEMISIMPLE TOPOLOGICAL PARTIAL \*-ALGEBRAS

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**ABSTRACT.** We continue our study of topological partial \*-algebras, focusing our attention to \*-semisimple partial \*-algebras, that is, those that possess a multiplication core and sufficiently many \*-representations. We discuss the respective roles of invariant positive sesquilinear (ips) forms and representable continuous linear functionals and focus on the case where the two notions are completely interchangeable (fully representable partial \*-algebras) with the scope of characterizing a \*-semisimple partial \*-algebra. Finally we describe various notions of bounded elements in such a partial \*-algebra, in particular, those defined in terms of a positive cone (order bounded elements). The outcome is that, for an appropriate order relation, one recovers the  $\mathcal{M}$ -bounded elements introduced in previous works.

## 1. INTRODUCTION

Studies on partial \*-algebras have provided so far a considerable amount of information about their representation theory and their structure. Many results have been obtained for concrete partial \*-algebras, i.e., partial \*-algebras of closable operators (the so-called partial O\*-algebras), but a substantial body of knowledge has been gathered also for abstract partial \*-algebras. A full analysis has been developed by Inoue and two of us some time ago and it can be found in the monograph [1], where earlier articles are quoted.

In a recent paper [4], we have started the analysis of certain types of bounded elements in a partial \*-algebra  $\mathfrak{A}$  and their incidence on the representation theory of  $\mathfrak{A}$ . It was shown, in particular, that the crucial condition is that  $\mathfrak{A}$  possesses sufficiently many invariant positive sesquilinear forms (ips-forms). The latter, in turn, generate \*-representations, that is, \*-homomorphisms into a partial O\*-algebra, via the well-known GNS construction. As in the particular case of a partial O\*-algebra, a spectral theory can then be developed, provided the partial \*-algebra has sufficiently many bounded elements. To that effect, we have introduced in [4] the notion of  $\mathcal{M}$ -bounded elements, associated to a sufficiently large family  $\mathcal{M}$  of ips-forms.

We continue this study in the present work, focusing on topological partial \*-algebras that possess what we call a *multiplication core*, that is, a

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dense subset of universal right multipliers with all the regularity properties necessary for a decent representation theory. In particular, we will require that our partial  $*$ -algebra has sufficiently many  $*$ -representations, a property usually characterized, for topological  $*$ -algebras, in terms of the so-called  *$*$ -radical*. When the latter is reduced to  $\{0\}$ , the partial  $*$ -algebra is called  *$*$ -semisimple*, the main subject of the paper. According to what we just said,  $*$ -semisimplicity is defined in terms of a family  $\mathcal{M}$  of ips-forms. Since it may be difficult to identify such a family in practice, we examine in what sense ips-forms may be replaced by a special class of continuous linear functionals, called *representable*. This leads to identify a class of topological partial  $*$ -algebras for which representable linear functionals and ips-forms can be freely replaced by one another, since every representable linear functional comes (as for  $*$ -algebras with unit) from an ips-form. These partial  $*$ -algebras are called *fully representable* (extending the analogous concept discussed in [7] for locally convex quasi  $*$ -algebras) and the interplay of this notion with  $*$ -semisimplicity is investigated.

This being done, we may come back to bounded elements of a  $*$ -semisimple partial  $*$ -algebra, more precisely to elements bounded with respect to some positive cone, thus defined in purely algebraic terms. Early work in that direction has been done by Vidav [13] and Schmüdgen [9], then generalized in our previous paper [4]. Here we consider several types of order on a partial  $*$ -algebra and analyze the corresponding notion of order bounded elements. The outcome is that, under appropriate conditions, the correct notion reduces to that of  $\mathcal{M}$ -bounded ones introduced in [4]. Therefore, when the partial  $*$ -algebra has sufficiently many such elements, the whole spectral theory developed in [3] and [4] can be recovered.

The paper is organized as follows. Section 2 is devoted to some preliminaries about partial  $*$ -algebras, taken mostly from [1] and [3, 4]. In addition, we introduce the notion of multiplication core and draw some consequences. We introduce in Section 3 the notion of  $*$ -semisimple partial  $*$ -algebra and discuss some of its properties. In Section 4, we compare the respective roles of ips-forms and representable linear functionals, with particular reference to fully representable partial  $*$ -algebras, and discuss the relationship of the latter notion with that of  $*$ -semisimple partial  $*$ -algebra. Finally, Section 5 is devoted to the various notions of bounded elements, from  $\mathcal{M}$ -bounded to order bounded ones.

## 2. PRELIMINARIES

The following preliminary definitions will be needed in the sequel. For more details we refer to [1, 8].

A partial  $*$ -algebra  $\mathfrak{A}$  is a complex vector space with conjugate linear involution  $*$  and a distributive partial multiplication  $\cdot$ , defined on a subset  $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ , satisfying the property that  $(x, y) \in \Gamma$  if, and only if,  $(y^*, x^*) \in \Gamma$  and  $(x \cdot y)^* = y^* \cdot x^*$ . From now on we will write simply  $xy$  instead of  $x \cdot y$  whenever  $(x, y) \in \Gamma$ . For every  $y \in \mathfrak{A}$ , the set of left (resp. right) multipliers of  $y$  is denoted by  $L(y)$  (resp.  $R(y)$ ), i.e.,  $L(y) = \{x \in \mathfrak{A} : (x, y) \in \Gamma\}$  (resp.

$R(y) = \{x \in \mathfrak{A} : (y, x) \in \Gamma\}$ . We denote by  $L\mathfrak{A}$  (resp.  $R\mathfrak{A}$ ) the space of universal left (resp. right) multipliers of  $\mathfrak{A}$ .

In general, a partial \*-algebra is not associative, but in several situations a weaker form of associativity holds. More precisely, we say that  $\mathfrak{A}$  is *semi-associative* if  $y \in R(x)$  implies  $yz \in R(x)$ , for every  $z \in R\mathfrak{A}$ , and

$$(xy)z = x(yz).$$

The partial \*-algebra  $\mathfrak{A}$  has a unit if there exists an element  $e \in \mathfrak{A}$  such that  $e = e^*$ ,  $e \in R\mathfrak{A} \cap L\mathfrak{A}$  and  $xe = ex = x$ , for every  $x \in \mathfrak{A}$ .

Let  $\mathcal{H}$  be a complex Hilbert space and  $\mathcal{D}$  a dense subspace of  $\mathcal{H}$ . We denote by  $L^\dagger(\mathcal{D}, \mathcal{H})$  the set of all (closable) linear operators  $X$  such that  $D(X) = \mathcal{D}$ ,  $D(X^*) \supseteq \mathcal{D}$ . The set  $L^\dagger(\mathcal{D}, \mathcal{H})$  is a partial \*-algebra with respect to the following operations: the usual sum  $X_1 + X_2$ , the scalar multiplication  $\lambda X$ , the involution  $X \mapsto X^\dagger := X^* \upharpoonright \mathcal{D}$  and the (weak) partial multiplication  $X_1 \square X_2 := X_1^{\dagger*} X_2$ , defined whenever  $X_2$  is a weak right multiplier of  $X_1$  (we shall write  $X_2 \in R^w(X_1)$  or  $X_1 \in L^w(X_2)$ ), that is, whenever  $X_2 \mathcal{D} \subset \mathcal{D}(X_1^{\dagger*})$  and  $X_1^* \mathcal{D} \subset \mathcal{D}(X_2^*)$ .

It is easy to check that  $X_1 \in L^w(X_2)$  if and only if there exists  $Z \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  such that

$$(1) \quad \langle X_2 \xi | X_1^\dagger \eta \rangle = \langle Z \xi | \eta \rangle, \quad \forall \xi, \eta \in \mathcal{D}.$$

In this case  $Z = X_1 \square X_2$ .  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  is neither associative nor semi-associative. If  $I$  denotes the identity operator of  $\mathcal{H}$ ,  $I_{\mathcal{D}} := I \upharpoonright \mathcal{D}$  is the unit of the partial \*-algebra  $L^\dagger(\mathcal{D}, \mathcal{H})$ .

If  $\mathfrak{N} \subseteq \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  we denote by  $R^w \mathfrak{N}$  the set of right multipliers of all elements of  $\mathfrak{N}$ . We recall that

$$(2) \quad R\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) \equiv R^w \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) = \{A \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) : A \text{ bounded and } A : \mathcal{D} \rightarrow \mathcal{D}^*\},$$

where

$$\mathcal{D}^* = \bigcap_{X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})} D(X^{\dagger*}).$$

We denote by  $L_b^\dagger(\mathcal{D}, \mathcal{H})$  the bounded part of  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ , i.e.,  $L_b^\dagger(\mathcal{D}, \mathcal{H}) = \{X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) : X \text{ is a bounded operator}\} = \{X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) : \overline{X} \in \mathcal{B}(\mathcal{H})\}$ .

A  $^\dagger$ -invariant subspace  $\mathfrak{M}$  of  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  is called a (weak) partial O\*-algebra if  $X \square Y \in \mathfrak{M}$ , for every  $X, Y \in \mathfrak{M}$  such that  $X \in L^w(Y)$ .  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  is the maximal partial O\*-algebra on  $\mathcal{D}$ .

The set  $\mathcal{L}^\dagger(\mathcal{D}) := \{X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) : X, X^\dagger : \mathcal{D} \rightarrow \mathcal{D}\}$  is a \*-algebra; more precisely, it is the maximal O\*-algebra on  $\mathcal{D}$  (for the theory of O\*-algebras and their representations we refer to [8]).

In the sequel, we will need the following topologies on  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ :

- The *strong topology*  $\mathfrak{t}_s$  on  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ , defined by the seminorms

$$p_\xi(X) = \|X\xi\|, \quad X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}), \quad \xi \in \mathcal{D}.$$

- The *strong\* topology*  $\mathfrak{t}_{s^*}$  on  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ , defined by the seminorms

$$p_\xi^*(X) = \max\{\|X\xi\|, \|X^\dagger\xi\|\}, \quad \xi \in \mathcal{D}.$$

A *\*-representation* of a partial \*-algebra  $\mathfrak{A}$  in the Hilbert space  $\mathcal{H}$  is a linear map  $\pi : \mathfrak{A} \rightarrow \mathcal{L}^\dagger(\mathcal{D}(\pi), \mathcal{H})$  such that: (i)  $\pi(x^*) = \pi(x)^\dagger$  for every  $x \in \mathfrak{A}$ ; (ii)  $x \in L(y)$  in  $\mathfrak{A}$  implies  $\pi(x) \in L^w(\pi(y))$  and  $\pi(x) \square \pi(y) = \pi(xy)$ . The subspace  $\mathcal{D}(\pi)$  is called the *domain* of the \*-representation  $\pi$ . The \*-representation  $\pi$  is said to be *bounded* if  $\overline{\pi(x)} \in \mathcal{B}(\mathcal{H})$  for every  $x \in \mathfrak{A}$ .

Let  $\varphi$  be a positive sesquilinear form on  $D(\varphi) \times D(\varphi)$ , where  $D(\varphi)$  is a subspace of  $\mathfrak{A}$ . Then we have

$$(3) \quad \varphi(x, y) = \overline{\varphi(y, x)}, \quad \forall x, y \in D(\varphi),$$

$$(4) \quad |\varphi(x, y)|^2 \leq \varphi(x, x)\varphi(y, y), \quad \forall x, y \in D(\varphi).$$

We put

$$N_\varphi = \{x \in D(\varphi) : \varphi(x, x) = 0\}.$$

By (4), we have

$$N_\varphi = \{x \in D(\varphi) : \varphi(x, y) = 0, \quad \forall y \in D(\varphi)\},$$

and so  $N_\varphi$  is a subspace of  $D(\varphi)$  and the quotient space  $D(\varphi)/N_\varphi := \{\lambda_\varphi(x) \equiv x + N_\varphi; x \in D(\varphi)\}$  is a pre-Hilbert space with respect to the inner product

$$\langle \lambda_\varphi(x) | \lambda_\varphi(y) \rangle = \varphi(x, y), \quad x, y \in D(\varphi).$$

We denote by  $\mathcal{H}_\varphi$  the Hilbert space obtained by completion of  $D(\varphi)/N_\varphi$ .

A positive sesquilinear form  $\varphi$  on  $\mathfrak{A} \times \mathfrak{A}$  is said to be *invariant*, and called an *ips-form*, if there exists a subspace  $B(\varphi)$  of  $\mathfrak{A}$  (called a *core* for  $\varphi$ ) with the properties

- (ips<sub>1</sub>)  $B(\varphi) \subset R\mathfrak{A}$ ;
- (ips<sub>2</sub>)  $\lambda_\varphi(B(\varphi))$  is dense in  $\mathcal{H}_\varphi$ ;
- (ips<sub>3</sub>)  $\varphi(xa, b) = \varphi(a, x^*b)$ ,  $\forall x \in \mathfrak{A}, \forall a, b \in B(\varphi)$ ;
- (ips<sub>4</sub>)  $\varphi(x^*a, yb) = \varphi(a, (xy)b)$ ,  $\forall x \in L(y), \forall a, b \in B(\varphi)$ .

In other words, an ips-form is an *everywhere defined* biweight, in the sense of [1].

To every ips-form  $\varphi$  on  $\mathfrak{A}$ , with core  $B(\varphi)$ , there corresponds a triple  $(\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ , where  $\mathcal{H}_\varphi$  is a Hilbert space,  $\lambda_\varphi$  is a linear map from  $B(\varphi)$  into  $\mathcal{H}_\varphi$  and  $\pi_\varphi$  is a \*-representation of  $\mathfrak{A}$  in the Hilbert space  $\mathcal{H}_\varphi$ . We refer to [1] for more details on this celebrated GNS construction.

Let  $\mathfrak{A}$  be a partial \*-algebra and  $\pi$  a \*-representation of  $\mathfrak{A}$  in  $\mathcal{D}(\pi)$ . For  $\xi \in \mathcal{D}(\pi)$  we put

$$(5) \quad \varphi_\pi^\xi(x, y) := \langle \pi(x)\xi | \pi(y)\xi \rangle, \quad x, y \in \mathfrak{A}.$$

Then,  $\varphi_\pi^\xi$  is a positive sesquilinear form on  $\mathfrak{A} \times \mathfrak{A}$ .

Let  $\mathfrak{B} \subseteq R\mathfrak{A}$  and assume that  $\pi(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))$ . Then it is easily seen that  $\varphi_\pi^\xi$  satisfies the conditions (ips<sub>3</sub>) and (ips<sub>4</sub>) above. However,  $\varphi_\pi^\xi$  is not necessarily an ips-form since  $\pi(\mathfrak{B})\xi$  may fail to be dense in  $\mathcal{H}$ . For this

reason, the following notion of *regular* \*-representation was introduced in [11].

**Definition 2.1.** A \*-representation  $\pi$  of  $\mathfrak{A}$  with domain  $\mathcal{D}(\pi)$  is called  *$\mathfrak{B}$ -regular* if  $\varphi_\pi^\xi$  is an ips-form with core  $\mathfrak{B}$ , for every  $\xi \in \mathcal{D}(\pi)$ .

**Remark 2.2.** The notion of regular \*-representation was given in [2] for a larger class of positive sesquilinear forms (biweights) referring to the *natural* core

$$B(\varphi_\pi^\xi) = \{a \in R\mathfrak{A} : \pi(a) \in \mathcal{D}^{**}(\pi)\}$$

(we refer to [1] for precise definitions). If  $\pi(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))$ , the  $\mathfrak{B}$ -regularity implies that  $\varphi_\pi^\xi$  is also ips-form with core  $B(\varphi_\pi^\xi)$ . We will come back to this point in Proposition 2.8.

Let  $\mathfrak{A}$  be a partial \*-algebra. We assume that  $\mathfrak{A}$  is a locally convex Hausdorff vector space under the topology  $\tau$  defined by a (directed) set  $\{p_\alpha\}_{\alpha \in \mathcal{I}}$  of seminorms. Assume that<sup>1</sup>

- (cl) for every  $x \in \mathfrak{A}$ , the linear map  $\mathsf{L}_x : R(x) \mapsto \mathfrak{A}$  with  $\mathsf{L}_x(y) = xy$ ,  $y \in R(x)$ , is closed with respect to  $\tau$ , in the sense that, if  $\{y_\alpha\} \subset R(x)$  is a net such that  $y_\alpha \rightarrow y$  and  $xy_\alpha \rightarrow z \in \mathfrak{A}$ , then  $y \in R(x)$  and  $z = xy$ .

For short, we will say that, in this case,  $\mathfrak{A}$  is a *topological partial \*-algebra*. If the involution  $x \mapsto x^*$  is continuous, we say that  $\mathfrak{A}$  is a *\*-topological partial \*-algebra*.

Starting from the family of seminorms  $\{p_\alpha\}_{\alpha \in \mathcal{I}}$ , we can define a second topology  $\tau^*$  on  $\mathfrak{A}$  by introducing the set of seminorms  $\{p_\alpha^*(x)\}$ , where

$$p_\alpha^*(x) = \max\{p_\alpha(x), p_\alpha(x^*)\}, \quad x \in \mathfrak{A}.$$

The involution  $x \mapsto x^*$  is automatically  $\tau^*$ -continuous. By (cl) it follows that, for every  $x \in \mathfrak{A}$ , both maps  $\mathsf{L}_x$ ,  $\mathsf{R}_x$  are  $\tau^*$ -closed. Hence,  $\mathfrak{A}[\tau^*]$  is a \*-topological partial \*-algebra.

In this paper we will consider the following particular classes of topological partial \*-algebras.

**Definition 2.3.** Let  $\mathfrak{A}[\tau]$  be a topological partial \*-algebra with locally convex topology  $\tau$ . Then,

- (1) A subspace  $\mathfrak{B}$  of  $R\mathfrak{A}$  is called a *multiplication core* if
  - (d<sub>1</sub>)  $e \in \mathfrak{B}$  if  $\mathfrak{A}$  has a unit  $e$ ;
  - (d<sub>2</sub>)  $\mathfrak{B} \cdot \mathfrak{B} \subseteq \mathfrak{B}$ ;
  - (d<sub>3</sub>)  $\mathfrak{B}$  is  $\tau^*$ -dense in  $\mathfrak{A}$ ;
  - (d<sub>4</sub>) for every  $b \in \mathfrak{B}$ , the map  $x \mapsto xb$ ,  $x \in \mathfrak{A}$ , is  $\tau$ -continuous;
  - (d<sub>5</sub>) one has  $b^*(xc) = (b^*x)c$ ,  $\forall x \in \mathfrak{A}, b, c \in \mathfrak{B}$ .
- (2)  $\mathfrak{A}[\tau]$  is called  *$\mathfrak{A}_0$ -regular* if it possesses<sup>2</sup> a multiplication core  $\mathfrak{A}_0$  which is a \*-algebra and, for every  $b \in \mathfrak{A}_0$ , the map  $x \mapsto bx$ ,  $x \in \mathfrak{A}$ , is  $\tau$ -continuous ([4, Def. 4.1]).

<sup>1</sup> Condition (cl) was called (t1) in [3].

<sup>2</sup> In [4] it was only supposed that  $\mathfrak{A}_0$  is  $\tau$ -dense in  $\mathfrak{A}$ .

If  $\mathfrak{A}$  is  $\mathfrak{A}_0$ -regular and if, in addition, the involution  $x \mapsto x^*$  is  $\tau$ -continuous for all  $x \in \mathfrak{A}$ , then the couple  $(\mathfrak{A}, \mathfrak{A}_0)$  is a locally convex *quasi  $*$ -algebra*.

**Remark 2.4.** A simple limiting argument shows that, if  $\mathfrak{B}$  is an algebra (i.e., it is also associative), then  $\mathfrak{A}$  is a  $\mathfrak{B}$ -right module, i.e.,

$$(xa)b = x(ab), \quad \forall x \in \mathfrak{A}, a, b \in \mathfrak{B}.$$

If  $\mathfrak{A}$  is  $\mathfrak{A}_0$ -regular then, in a similar way,

$$(xa)b = x(ab), \quad (ax)b = a(xb) \text{ for every } x \in \mathfrak{A}, a, b \in \mathfrak{A}_0.$$

**Remark 2.5.** We warn the reader that an  $\mathfrak{A}_0$ -regular topological partial  $*$ -algebra  $\mathfrak{A}[\tau]$  is not necessarily a locally convex partial  $*$ -algebra in the sense of [1, Def. 2.1.8]. Neither need it be topologically regular in the sense of [4, Def. 2.1.8], which is a more restrictive notion.

**Remark 2.6.** Let  $\mathfrak{A}[\tau]$  be an  $\mathfrak{A}_0$ -regular topological partial  $*$ -algebra. Then, for every  $b \in \mathfrak{A}_0$ , the maps  $x \mapsto xb$  and  $x \mapsto bx$ ,  $x \in \mathfrak{A}$ , are also  $\tau^*$ -continuous. However, the density of  $\mathfrak{A}_0$  in  $\mathfrak{A}[\tau^*]$  may fail. Thus  $\mathfrak{A}[\tau^*]$  need not be an  $\mathfrak{A}_0$ -regular  $*$ -topological partial  $*$ -algebra.

**Examples 2.7.** The three notions given in Definition 2.3 are really different.

(1) Take  $\mathfrak{A} = \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ . Then,  $R\mathfrak{A}$  is given in (2), so that we have an example where  $R\mathfrak{A} \cdot R\mathfrak{A} \not\subset R\mathfrak{A}$ .

(2) Take again  $\mathfrak{A} = \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})[\mathfrak{t}_{s*}]$ . Then  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  is  $\mathfrak{A}_0$ -regular for  $\mathfrak{A}_0 = L_b^\dagger(\mathcal{D})$ .

(3) Assume  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})[\mathfrak{t}_{s*}]$  is self-adjoint, i.e.  $\mathcal{D} = \mathcal{D}^*$  (for instance, when  $\mathcal{D} = D^\infty(A)$  for a self-adjoint operator  $A$ ). Then  $R\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) = \{X \in L_b^\dagger(\mathcal{D}, \mathcal{H}) : X : \mathcal{D} \rightarrow \mathcal{D}\}$  is an algebra, but it is not  $*$ -invariant. Hence it is a multiplication core, since it is  $\mathfrak{t}_{s*}$ -dense in  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ , but  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  is not  $R\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ -regular.

The case of a locally convex quasi  $*$ -algebra  $(\mathfrak{A}, \mathfrak{A}_0)$  was studied in [7] and a number of interesting properties have been derived. Some of these extend to the general case of a partial  $*$ -algebra, as we shall see in the sequel.

**Proposition 2.8.** *Let  $\mathfrak{A}[\tau]$  be a topological partial  $*$ -algebra and  $\mathfrak{B}$  a multiplication core. Then every  $(\tau, \mathfrak{t}_s)$ -continuous  $*$ -representation of  $\mathfrak{A}$  is  $\mathfrak{B}$ -regular.*

*Proof.* First we may assume that  $\pi(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))$ . Indeed, put

$$D(\pi_1) := \left\{ \xi_0 + \sum_{i=1}^n \pi(b_i)\xi_i : b_i \in \mathfrak{B}, \xi_i \in \mathcal{D}(\pi); i = 0, 1, \dots, n \right\},$$

$$\pi_1(x) \left( \xi_0 + \sum_{i=1}^n \pi(b_i)\xi_i \right) := \pi(x)\xi_0 + \sum_{i=1}^n (\pi(x) \square \pi(b_i))\xi_i.$$

Then, exactly as in [4] we can prove that  $\pi_1$  is a  $*$ -representation of  $\mathfrak{A}$  with  $\pi_1(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi_1))$ .

If  $\pi$  is  $(\tau, \mathfrak{t}_s)$ -continuous, then  $\pi_1$  is  $(\tau, \mathfrak{t}_s)$ -continuous too (recall that domains are different!). Indeed, if  $x_\alpha \xrightarrow{\tau} x$ , then  $\pi(x_\alpha)\xi \rightarrow \pi(x)\xi$ , for every

$\xi \in \mathcal{D}(\pi)$ . The continuity of the right multiplication then implies that  $x_\alpha b \xrightarrow{\tau} xb$ , for every  $b \in \mathfrak{B}$ . Thus, by the continuity of  $\pi$ , we get, for every  $b \in \mathfrak{B}$ ,  $\pi(x_\alpha b)\xi \rightarrow \pi(xb)\xi$ , for every  $\xi \in \mathcal{D}(\pi)$  or, equivalently,  $(\pi(x_\alpha) \square \pi(b))\xi \rightarrow (\pi(x) \square \pi(b))\xi$ , for every  $\xi \in \mathcal{D}(\pi)$ . Hence

$$\begin{aligned} \pi_1(x_\alpha) \left( \sum_{i=1}^n \pi(b_i)\xi_i \right) &= \sum_{i=1}^n (\pi(x_\alpha) \square \pi(b_i))\xi_i \rightarrow \\ &\sum_{i=1}^n (\pi(x) \square \pi(b_i))\xi_i = \pi_1(x) \left( \sum_{i=1}^n \pi(b_i)\xi_i \right). \end{aligned}$$

Thus, every  $(\tau, \mathfrak{t}_s)$ -continuous \*-representation  $\pi$  extends to a  $(\tau, \mathfrak{t}_s)$ -continuous \*-representation  $\pi_1$  with  $\pi_1(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi_1))$ . Finally we prove the  $\mathfrak{B}$ -regularity of  $\pi$ . If  $x \in \mathfrak{A}$  then there exists a net  $\{b_\alpha\} \subset \mathfrak{B}$  such that  $b_\alpha \xrightarrow{\tau} x$ . Then we have

$$\|\lambda_{\varphi_\pi^\xi}(x) - \lambda_{\varphi_\pi^\xi}(b_\alpha)\|^2 = \varphi_\pi^\xi(x - b_\alpha, x - b_\alpha) \leq p(x - b_\alpha)^2 \rightarrow 0,$$

where  $p$  is a convenient  $\tau$ -continuous seminorm. This implies that  $\lambda_{\varphi_\pi^\xi}(\mathfrak{B})$  is dense in  $\mathcal{H}_{\varphi_\pi^\xi}$ . Hence  $\varphi_\pi^\xi$  is an ips-form with core  $\mathfrak{B}$ .  $\square$

Let  $\mathfrak{A}[\tau]$  be a topological partial \*-algebra with multiplication core  $\mathfrak{B}$  and  $\varphi$  a positive sesquilinear forms on  $\mathfrak{A} \times \mathfrak{A}$  for which the conditions (ips<sub>1</sub>), (ips<sub>3</sub>) and (ips<sub>4</sub>) are satisfied (with respect to  $\mathfrak{B}$ ). Suppose that  $\varphi$  is  $\tau$ -continuous, i.e., there exist  $p_\alpha, \gamma > 0$  such that:

$$|\varphi(x, y)| \leq \gamma p_\alpha(x) p_\alpha(y) \quad \forall x, y \in \mathfrak{A}.$$

Then (ips<sub>2</sub>) is also satisfied and, therefore,  $\mathfrak{B}$  is a core for  $\varphi$ , so that  $\varphi$  is an ips-form. We denote by  $\mathcal{P}_\mathfrak{B}(\mathfrak{A})$  the set of all  $\tau$ -continuous ips-forms with core  $\mathfrak{B}$ .

Using the continuity of the multiplication and Remark 2.4, it is easily seen that if  $\varphi \in \mathcal{P}_\mathfrak{B}(\mathfrak{A})$  and  $a \in \mathfrak{B}$ , then  $\varphi_a \in \mathcal{P}_\mathfrak{B}(\mathfrak{A})$ , where

$$\varphi_a(x, y) := \varphi(xa, ya), \quad x, y \in \mathfrak{A}.$$

### 3. TOPOLOGICAL PARTIAL \*-ALGEBRAS WITH SUFFICIENTLY MANY \*-REPRESENTATIONS

Throughout this paper we will be mostly concerned with topological partial \*-algebras possessing sufficiently many continuous \*-representations. In the case of topological \*-algebras this situation can be studied by introducing the so-called (topological) \*-radical of the algebra. Thus we extend this notion to topological partial \*-algebras.

Let  $\mathfrak{A}[\tau]$  be a topological partial \*-algebra. We define the *\*-radical* of  $\mathfrak{A}$  as

$$\mathcal{R}^*(\mathfrak{A}) := \{x \in \mathfrak{A} : \pi(x) = 0, \text{ for all } (\tau, \mathfrak{t}_s)\text{-continuous } *- \text{representations } \pi\}.$$

We put  $\mathcal{R}^*(\mathfrak{A}) = \mathfrak{A}$ , if  $\mathfrak{A}[\tau]$  has no  $(\tau, \mathfrak{t}_s)$ -continuous \*-representations.



**Remark 3.1.** The  $*$ -radical was defined in [4, Sec.5] as

$$\mathcal{R}_*(\mathfrak{A}) := \{x \in \mathfrak{A} : \pi(x) = 0, \text{ for all } (\tau, \mathfrak{t}_{s*})\text{-continuous } *\text{-representations } \pi\}.$$

However, the two definitions are equivalent. Indeed, since every  $(\tau, \mathfrak{t}_{s*})$ -continuous  $*$ -representation is  $(\tau, \mathfrak{t}_s)$ -continuous, we have  $\mathcal{R}_*(\mathfrak{A}) \subset \mathcal{R}^*(\mathfrak{A})$ . In order to prove that  $\mathcal{R}^*(\mathfrak{A}) \subset \mathcal{R}_*(\mathfrak{A})$ , assume that  $x \notin \mathcal{R}_*(\mathfrak{A})$ , i.e., there is a  $(\tau, \mathfrak{t}_{s*})$ -continuous  $*$ -representation  $\pi$  such that  $\pi(x) \neq 0$ . But  $\pi$  is also  $(\tau, \mathfrak{t}_s)$ -continuous, hence  $x \notin \mathcal{R}^*(\mathfrak{A})$  as well.

The  $*$ -radical enjoys the following immediate properties:

- (1) If  $x \in \mathcal{R}^*(\mathfrak{A})$ , then  $x^* \in \mathcal{R}^*(\mathfrak{A})$ .
- (2) If  $x \in \mathfrak{A}$ ,  $y \in \mathcal{R}^*(\mathfrak{A})$  and  $x \in L(y)$ , then  $xy \in \mathcal{R}^*(\mathfrak{A})$ .

From now on, we denote by  $\text{Rep}_c(\mathfrak{A})$  the set of all  $(\tau, \mathfrak{t}_s)$ -continuous  $*$ -representations of  $\mathfrak{A}$ . If  $\mathfrak{A}$  has a multiplication core  $\mathfrak{B}$ , we may always suppose that  $\pi(x) \in \mathcal{L}^\dagger(\mathcal{D}(\pi))$  for every  $x \in \mathfrak{B}$ , as results from the proof of Proposition 2.8.

**Proposition 3.2.** *Let  $\mathfrak{A}[\tau]$  be a topological partial  $*$ -algebra with unit  $e$ . Let  $\mathfrak{B}$  be a multiplication core. For an element  $x \in \mathfrak{A}$  the following statements are equivalent.*

- (i)  $x \in \mathcal{R}^*(\mathfrak{A})$ .
- (ii)  $\varphi(x, x) = 0$  for every  $\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ .

*Proof.* (i)  $\Rightarrow$  (ii): Let  $\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$  and  $\pi_\varphi$  the corresponding GNS representation. Then, for every  $x \in \mathfrak{A}$ ,

$$\|\pi_\varphi(x)\lambda_\varphi(a)\|^2 = \varphi(xa, xa) = \varphi_a(x, x) \leq p(x)^2, \quad a \in \mathfrak{B}$$

for some continuous  $\tau$ -seminorm  $p$  (depending on  $a$ ). Hence  $\pi_\varphi$  is  $(\tau, \mathfrak{t}_s)$ -continuous. If  $x \in \mathcal{R}^*(\mathfrak{A})$ , then  $\pi_\varphi(x) = 0$ . Thus  $\varphi(xa, xa) = 0$ , for every  $a \in \mathfrak{B}$ . From  $e \in \mathfrak{B}$ , we get the statement.

(ii)  $\Rightarrow$  (i) Let  $\pi \in \text{Rep}_c(\mathfrak{A})$ . We assume  $\pi(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))$ . For  $x, y \in \mathfrak{A}$  and  $\xi \in \mathcal{D}$ , put, as before,

$$\varphi_\pi^\xi(x, y) := \langle \pi(x)\xi | \pi(y)\xi \rangle, \quad x, y \in \mathfrak{A}.$$

Then,

$$|\varphi_\pi^\xi(x, y)| = |\langle \pi(x)\xi | \pi(y)\xi \rangle| \leq \|\pi(x)\xi\| \|\pi(y)\xi\| \leq p(x)p(y)$$

for some  $\tau$ -continuous seminorm  $p$ . Hence,  $\varphi_\pi^\xi$  is continuous.

Thus,  $\varphi_\pi^\xi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$  and, by the assumption,  $\|\pi(x)\xi\|^2 = \varphi_\pi^\xi(x, x) = 0$ . The arbitrariness of  $\xi$  implies that  $\pi(x) = 0$ .  $\square$

As for topological  $*$ -algebras, the  $*$ -radical contains all elements  $x$  whose square  $x^*x$  (if well defined) vanishes.

**Proposition 3.3.** *Let  $\mathfrak{A}$  be a topological partial  $*$ -algebra. Let  $x \in \mathfrak{A}$ , with  $x^* \in L(x)$ . If  $x^*x = 0$ , then  $x \in \mathcal{R}^*(\mathfrak{A})$*



*Proof.* If  $\pi$  is a  $(\tau, \mathbf{t}_s)$ -continuous \*-representation of  $\mathfrak{A}$ ,  $\pi(x^*) \square \pi(x) = \pi(x)^\dagger \square \pi(x)$  is well-defined and equals 0. Hence, for every  $\xi \in \mathcal{D}(\pi)$ ,

$$\begin{aligned} \|\pi(x)\xi\|^2 &= \langle \pi(x)\xi | \pi(x)\xi \rangle \\ &= \langle \pi(x)^\dagger \square \pi(x)\xi | \xi \rangle = \langle \pi(x^*) \square \pi(x)\xi | \xi \rangle \\ &= \langle \pi(x^*x)\xi | \xi \rangle = 0. \end{aligned}$$

Hence  $\pi(x) = 0$ .  $\square$

**Remark 3.4.** A sort of converse of the previous statement was stated in [4, Proposition 5.3]. Unfortunately, the proof given there contains a gap.

**Definition 3.5.** A topological partial \*-algebra  $\mathfrak{A}[\tau]$  is called *\*-semisimple* if, for every  $x \in \mathfrak{A} \setminus \{0\}$  there exists a  $(\tau, \mathbf{t}_s)$ -continuous \*-representation  $\pi$  of  $\mathfrak{A}$  such that  $\pi(x) \neq 0$  or, equivalently, if  $\mathcal{R}^*(\mathfrak{A}) = \{0\}$ .

By Proposition 3.2,  $\mathfrak{A}[\tau]$  is \*-semisimple if, and only if, for some multiplication core  $\mathfrak{B}$ , the family of ips-forms  $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$  is *sufficient* in the following sense [4].

**Definition 3.6.** A family  $\mathcal{M}$  of continuous ips-forms on  $\mathfrak{A} \times \mathfrak{A}$  is *sufficient* if  $x \in \mathfrak{A}$  and  $\varphi(x, x) = 0$  for every  $\varphi \in \mathcal{M}$  imply  $x = 0$ .

The sufficiency of the family  $\mathcal{M}$  can be described in several different ways.

**Lemma 3.7.** *Let  $\mathfrak{A}$  be a topological partial \*-algebra with multiplication core  $\mathfrak{B}$ . Then the following statements are equivalent:*

- (i)  $\mathcal{M}$  is sufficient.
- (ii)  $\varphi(xa, b) = 0$ , for every  $\varphi \in \mathcal{M}$  and  $a, b \in \mathfrak{B}$ , implies  $x = 0$ .
- (iii)  $\varphi(xa, a) = 0$ , for every  $\varphi \in \mathcal{M}$  and  $a \in \mathfrak{B}$ , implies  $x = 0$ .
- (iv)  $\varphi(xa, y) = 0$ , for every  $\varphi \in \mathcal{M}$  and  $y \in \mathfrak{A}$ ,  $a \in \mathfrak{B}$ , implies  $x = 0$ .
- (v)  $\varphi(xa, xa) = 0$  for every  $\varphi \in \mathcal{M}$  and  $a \in \mathfrak{B}$ , implies  $x = 0$ .

We omit the easy proof.

Of course, if the family  $\mathcal{M}$  is sufficient, any larger family  $\mathcal{M}' \supset \mathcal{M}$  is also sufficient. In this case, the maximal sufficient family (having  $\mathfrak{B}$  as core) is obviously the set  $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$  of *all* continuous ips-forms with core  $\mathfrak{B}$ . Hence if a sufficient family  $\mathcal{M} \subseteq \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$  exists,  $\mathfrak{A}[\tau]$  is \*-semisimple.

**Example 3.8.** As mentioned before, the space  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  is a  $\mathbf{L}_b^\dagger(\mathcal{D})$ -regular partial \*-algebra, when endowed with the strong\*-topology  $\mathbf{t}_{s^*}$ . The set of positive sesquilinear forms  $\mathcal{M} := \{\varphi_\xi : \xi \in \mathcal{D}\}$ , where  $\varphi_\xi(X, Y) = \langle X\xi | Y\xi \rangle$ ,  $X, Y \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ , is a sufficient family of ips-forms with core  $\mathbf{L}_b^\dagger(\mathcal{D})$ . Indeed, if  $\varphi_\xi(X, X) = 0$ , for every  $\xi \in \mathcal{D}$ , then  $\|X\xi\|^2 = 0$  and therefore  $X = 0$ .

**Example 3.9.** As shown in [6], the space  $L^p(X)$ ,  $X = [0, 1]$ , endowed with its usual norm topology, is  $L^\infty(X)$ -regular and it is \*-semisimple if  $p \geq 2$ . Indeed, in this case the family of all continuous ips-forms is given by  $\mathcal{M} = \{\varphi_w : w \in L^{p/(p-2)}, w \geq 0\}$ , where

$$\varphi_w(f, g) = \int_X f(t) \overline{g(t)} w(t) dt, \quad f, g \in L^p(X),$$

and it is sufficient.

If  $1 \leq p < 2$ , the set of all continuous ips-forms reduces to  $\{0\}$ . Hence, in this case,  $\mathcal{R}^*(L^p(X)) = L^p(X)$ .

Let  $\mathfrak{A}$  be a topological partial  $*$ -algebra with multiplication core  $\mathfrak{B}$ . If  $\mathfrak{A}$  possesses a sufficient family  $\mathcal{M}$  of ips-forms, an *extension* of the multiplication of  $\mathfrak{A}$  can be introduced in a way similar to [4, Sec.4].

We say that the *weak* multiplication  $x \boxdot y$  is well-defined (with respect to  $\mathcal{M}$ ) if there exists  $z \in \mathfrak{A}$  such that:

$$\varphi(ya, x^*b) = \varphi(za, b), \quad \forall a, b \in \mathfrak{B}, \forall \varphi \in \mathcal{M}.$$

In this case, we put  $x \boxdot y := z$  and the sufficiency of  $\mathcal{M}$  guarantees that  $z$  is unique. The weak multiplication  $\boxdot$  clearly depends on  $\mathcal{M}$ : the larger is  $\mathcal{M}$ , the stronger is the weak multiplication, in the sense that if  $\mathcal{M} \subseteq \mathcal{M}' \subseteq \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$  and  $x \boxdot y$  exists w.r. to  $\mathcal{M}'$ , then  $x \boxdot y$  exists with respect to  $\mathcal{M}$  too.

A handy criterion for the existence of the weak multiplication is provided by the following

**Proposition 3.10.** *Let  $\mathfrak{B}$  be an algebra, then the weak product  $x \boxdot y$  is defined (with respect to  $\mathcal{M}$ ) if, and only if, there exists a net  $\{b_\alpha\}$  in  $\mathfrak{B}$  such that  $b_\alpha \xrightarrow{\tau} y$  and  $xb_\alpha \xrightarrow{\tau_w^{\mathcal{M}}} z \in \mathfrak{A}$ .*

Here  $\tau_w^{\mathcal{M}}$  is the weak topology determined by  $\mathcal{M}$ , with seminorms  $x \mapsto |\varphi(xa, b)|$ ,  $\varphi \in \mathcal{M}, a, b \in \mathfrak{B}$ . It is easy to prove that  $\mathfrak{A}$  is also a partial  $*$ -algebra with respect to the weak multiplication.

Since it holds in typical examples, e.g.  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  [4, Prop. 3.2], we will often suppose that the following condition is satisfied:

(wp)  $xy$  exists if, and only if,  $x \boxdot y$  exists. In this case  $xy = x \boxdot y$ .

In this situation it is possible to define a stronger multiplication on  $\mathfrak{A}$ : we say that the *strong* multiplication  $x \bullet y$  is well-defined (and that  $x \in L^s(y)$  or  $y \in R^s(x)$ ) if  $x \in L(y)$  and:

- (sm<sub>1</sub>)  $\varphi((xy)a, z^*b) = \varphi(ya, (x^*z^*)b), \quad \forall z \in L(x), \forall \varphi \in \mathcal{M}, \forall a, b \in \mathfrak{B};$
- (sm<sub>2</sub>)  $\varphi((y^*x^*)a, vb) = \varphi(x^*a, (yv)b), \quad \forall v \in R(y), \forall \varphi \in \mathcal{M}, \forall a, b \in \mathfrak{B}.$

The same considerations on the dependence on  $\mathcal{M}$  of the weak multiplication apply, of course, to the strong multiplication.

**Definition 3.11.** Let  $\mathfrak{A}$  be a partial  $*$ -algebra. A  $*$ -representation  $\pi$  of  $\mathfrak{A}$  is called *quasi-symmetric* if, for every  $x \in \mathfrak{A}$ ,

$$\begin{aligned} \bigcap_{z \in L(x)} D((\pi(x)^* \upharpoonright \pi(z^*)\mathcal{D}(\pi))^* &= D(\overline{\pi(x)}); \\ \bigcap_{v \in R(x)} D((\pi(x^*)^* \upharpoonright \pi(v)\mathcal{D}(\pi))^* &= D(\overline{\pi(x)^\dagger}). \end{aligned}$$

Of course, the same definition can be given for any partial  $O^*$ -algebra  $\mathfrak{M}$  (by considering the identical  $*$ -representation).

**Remark 3.12.** The conditions given in the previous Definition are certainly satisfied if, for every  $x \in \mathfrak{A}$ , there exist  $s \in L(x), t \in R(x)$  such that

$(\pi(x)^* \upharpoonright \pi(s^*)\mathcal{D}(\pi))^* = \overline{\pi(x)}$  and  $(\pi(x^*)^* \upharpoonright \pi(t)\mathcal{D}(\pi))^* = \overline{\pi(x^*)}$ . These stronger conditions are satisfied, in particular, by any symmetric O\*-algebra  $\mathfrak{M}$  (*symmetric* means that  $(I + X^*\overline{X})^{-1}$  is in the bounded part of  $\mathfrak{M}$ , for every  $X \in \mathfrak{M}$ ). The proof given in [4, Theorem 3.5], shows that in this case  $\mathfrak{M}$  is quasi-symmetric, whence the name. In particular,  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  is quasi-symmetric.

**Proposition 3.13.** *Let  $\mathfrak{A}$  be a \*-semisimple topological partial \*-algebra with multiplication core  $\mathfrak{B}$  containing the unit  $e$  of  $\mathfrak{A}$ . If the strong product  $x \bullet y$  of  $x, y \in \mathfrak{A}$  is well-defined with respect to  $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ , then for every quasi-symmetric  $\pi \in \text{Rep}_c(\mathfrak{A}, \mathfrak{B})$  with  $\pi(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))$  and  $\pi(e) = I_{\mathcal{D}(\pi)}$ , the strong product  $\pi(x) \circ \pi(y)$  is well-defined too and*

$$\pi(x \bullet y) = \pi(x) \circ \pi(y).$$

*Proof.* Let  $\pi \in \text{Rep}_c(\mathfrak{A}, \mathfrak{B})$  satisfy the required assumptions. Let  $\xi \in \mathcal{D}(\pi)$  and define  $\varphi_\pi^\xi$  as in the proof of Proposition 3.2. Then  $\varphi_\pi^\xi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$  and since  $x \bullet y$  is well-defined (with respect to  $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ ), we have

$$\begin{aligned} \varphi_\pi^\xi((xy)a, z^*b) &= \varphi_\pi^\xi(ya, (x^*z^*)b), \quad \forall z \in L(x), \forall a, b \in \mathfrak{B}; \\ \varphi_\pi^\xi((y^*x^*)a, vb) &= \varphi_\pi^\xi(x^*a, (yv)b), \quad \forall v \in R(y), \forall a, b \in \mathfrak{B}. \end{aligned}$$

Equivalently,

$$\begin{aligned} \langle (\pi(x) \square \pi(y))\pi(a)\xi | \pi(z^*)\pi(b)\xi \rangle &= \langle \pi(y)\pi(a)\xi | (\pi(x^*) \square \pi(z^*))\pi(b)\xi \rangle, \\ &\quad \forall z \in L(x), \forall a, b \in \mathfrak{B}; \\ \langle (\pi(y^*) \square \pi(x^*))\pi(a)\xi | \pi(v)\pi(b)\xi \rangle &= \langle \pi(x^*)\pi(a)\xi | (\pi(y) \square \pi(v))\pi(b)\xi \rangle, \\ &\quad \forall v \in R(y), \forall a, b \in \mathfrak{B}. \end{aligned}$$

By taking  $a = b = e$  and using the polarization identity, one gets, for every  $\xi, \eta \in \mathcal{D}(\pi)$ ,

$$\begin{aligned} \langle (\pi(x) \square \pi(y))\xi | \pi(z^*)\eta \rangle &= \langle \pi(y)\xi | (\pi(x^*) \square \pi(z^*))\eta \rangle, \quad \forall z \in L(x); \\ \langle (\pi(y^*) \square \pi(x^*))\xi | \pi(v)\eta \rangle &= \langle \pi(x^*)\xi | (\pi(y) \square \pi(v))\eta \rangle, \quad \forall v \in R(y). \end{aligned}$$

From these relations, it follows that

$$\begin{aligned} \pi(y) : \mathcal{D}(\pi) &\rightarrow D((\pi(x)^* \upharpoonright \pi(z^*)\mathcal{D}(\pi))^*), \quad \forall z \in L(x), \\ \pi(x^*) : \mathcal{D}(\pi) &\rightarrow D((\pi(y^*)^* \upharpoonright \pi(v)\mathcal{D}(\pi))^*), \quad \forall v \in R(y). \end{aligned}$$

By the assumption, it follows that  $\pi(y) : \mathcal{D}(\pi) \rightarrow D(\overline{\pi(x)})$  and  $\pi(x^*) : \mathcal{D}(\pi) \rightarrow D(\overline{\pi(y^*)})$ . Thus,  $\pi(x) \circ \pi(y)$  is well defined.  $\square$

#### 4. REPRESENTABLE FUNCTIONALS VERSUS IPS-FORMS

So far we have used ips-forms in order to characterize \*-semisimplicity of a topological partial \*-algebra. The reason lies in the fact that ips-forms allow a GNS-like construction. However, from a general point of view it is not easy to find conditions for the existence of sufficient families of ips-forms, whereas there exist well-known criteria for the existence of continuous *linear* functionals that separate points of  $\mathfrak{A}$ . However continuous linear functionals, which are positive in a certain sense, do not give rise, in general to a GNS

construction. This can be done, if they are *representable* [5] in the sense specified below. It is then natural to consider, in more details, conditions for the representability of continuous positive linear functionals.

**Definition 4.1.** Let  $\mathfrak{A}[\tau]$  be a topological partial  $*$ -algebra with multiplication core  $\mathfrak{B}$ . A continuous linear functional  $\omega$  on  $\mathfrak{A}$  is  $\mathfrak{B}$ -positive if  $\omega(a^*a) \geq 0$  for every  $a \in \mathfrak{B}$ .

The continuity of  $\omega$  implies that  $\omega(x) \geq 0$  for every  $x$  which belongs to the  $\tau$ -closure  $\mathfrak{A}^+(\mathfrak{B})$  of the set

$$\mathfrak{B}^{(2)} = \left\{ \sum_{k=1}^n x_k^* x_k, x_k \in \mathfrak{B}, n \in \mathbb{N} \right\}.$$

In the very same way as in [7, Theorem 3.2] one can prove the following

**Theorem 4.2.** Assume that  $\mathfrak{A}^+(\mathfrak{B}) \cap (-\mathfrak{A}^+(\mathfrak{B})) = \{0\}$ . Let  $a \in \mathfrak{A}^+(\mathfrak{B})$ ,  $a \neq 0$ . Then there exists a continuous linear functional  $\omega$  on  $\mathfrak{A}$  with the properties:

- (i)  $\omega(x) \geq 0, \forall x \in \mathfrak{A}^+(\mathfrak{B});$
- (ii)  $\omega(a) > 0.$

The set  $\mathfrak{A}^+(\mathfrak{B})$  will play an important role in Theorem 4.12 and in the analysis of order bounded elements in Section 5.2.

**Definition 4.3.** Let  $\omega$  be a linear functional on  $\mathfrak{A}$  and  $\mathfrak{B}$  a subspace of  $R\mathfrak{A}$ . We say that  $\omega$  is *representable* (with respect to  $\mathfrak{B}$ ) if the following requirements are satisfied:

- (r<sub>1</sub>)  $\omega(a^*a) \geq 0$  for all  $a \in \mathfrak{B}$ ;
- (r<sub>2</sub>)  $\omega(b^*(x^*a)) = \overline{\omega(a^*(xb))}, \forall a, b \in \mathfrak{B}, x \in \mathfrak{A}$ ;
- (r<sub>3</sub>)  $\forall x \in \mathfrak{A}$  there exists  $\gamma_x > 0$  such that  $|\omega(x^*a)| \leq \gamma_x \omega(a^*a)^{1/2}$ , for all  $a \in \mathfrak{B}$ .

In this case, one can prove that there exists a triple  $(\pi_\omega^\mathfrak{B}, \lambda_\omega^\mathfrak{B}, \mathcal{H}_\omega^\mathfrak{B})$  such that

- (a)  $\pi_\omega^\mathfrak{B}$  is a  $*$ -representation of  $\mathfrak{A}$  in  $\mathcal{H}_\omega$ ;
- (b)  $\lambda_\omega^\mathfrak{B}$  is a linear map of  $\mathfrak{A}$  into  $\mathcal{H}_\omega^\mathfrak{B}$  with  $\lambda_\omega^\mathfrak{B}(\mathfrak{B}) = \mathcal{D}(\pi_\omega^\mathfrak{B})$  and  $\pi_\omega^\mathfrak{B}(x)\lambda_\omega^\mathfrak{B}(a) = \lambda_\omega^\mathfrak{B}(xa)$ , for every  $x \in \mathfrak{A}, a \in \mathfrak{B}$ .
- (c)  $\omega(b^*(xa)) = \langle \pi_\omega^\mathfrak{B}(x)\lambda_\omega^\mathfrak{B}(a) | \lambda_\omega^\mathfrak{B}(b) \rangle$ , for every  $x \in \mathfrak{A}, a, b \in \mathfrak{B}$ .

In particular, if  $\mathfrak{A}$  has a unit  $e$  and  $e \in \mathfrak{B}$ , we have:

- (a<sub>1</sub>)  $\pi_\omega^\mathfrak{B}$  is a cyclic  $*$ -representation of  $\mathfrak{A}$  with cyclic vector  $\xi_\omega$ ;
- (b<sub>1</sub>)  $\lambda_\omega^\mathfrak{B}$  is a linear map of  $\mathfrak{A}$  into  $\mathcal{H}_\omega^\mathfrak{B}$  with  $\lambda_\omega^\mathfrak{B}(\mathfrak{B}) = \mathcal{D}(\pi_\omega^\mathfrak{B})$ ,  $\xi_\omega = \lambda_\omega^\mathfrak{B}(e)$  and  $\pi_\omega^\mathfrak{B}(x)\lambda_\omega^\mathfrak{B}(a) = \lambda_\omega^\mathfrak{B}(xa)$ , for every  $x \in \mathfrak{A}, a \in \mathfrak{B}$ .
- (c<sub>1</sub>)  $\omega(a) = \langle \pi_\omega^\mathfrak{B}(x)\xi_\omega | \xi_\omega \rangle$ , for every  $x \in \mathfrak{A}$ .

The GNS construction then depends on the subspace  $\mathfrak{B}$ . We adopt the notation  $\mathcal{R}(\mathfrak{A}, \mathfrak{B})$  for denoting the set of linear functionals on  $\mathfrak{A}$  which are representable with respect to the same  $\mathfrak{B}$ .

**Remark 4.4.** It is worth recalling (also for fixing notations) that the Hilbert space  $\mathcal{H}_\omega$  is defined by considering the subspace of  $\mathfrak{B}$

$$N_\omega = \{x \in \mathfrak{B}; \omega(y^*x) = 0, \forall y \in \mathfrak{B}\}.$$

The quotient  $\mathfrak{B}/N_\omega \equiv \{\lambda_\omega^0(x) := x + N_\omega; x \in \mathfrak{B}\}$  is a pre-Hilbert space with inner product

$$\langle \lambda_\omega^0(x) | \lambda_\omega^0(y) \rangle = \omega(y^*x), \quad x, y \in \mathfrak{B}.$$

Then  $\mathcal{H}_\omega$  is the completion of  $\lambda_\omega^0(\mathfrak{B})$ . The representability of  $\omega$  implies that  $\lambda_\omega^0 : \mathfrak{B} \rightarrow \mathcal{H}_\omega$  extends to a linear map  $\lambda_\omega : \mathfrak{A} \rightarrow \mathcal{H}_\omega$ .

**Remark 4.5.** We notice that if  $\pi$  is a \*-representation of  $\mathfrak{A}$  on the domain  $\mathcal{D}(\pi)$ , and  $\mathfrak{B}$  is a subspace of  $R\mathfrak{A}$  such that  $\pi(\mathfrak{B}) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))$ , then, for every  $\xi \in \mathcal{D}(\pi)$ , the linear functional  $\omega_\pi^\xi$  defined by  $\omega_\pi^\xi(x) = \langle \pi(x)\xi | \xi \rangle$  is representable, whereas the corresponding sesquilinear form  $\varphi_\pi^\xi(x, y) = \langle \pi(x)\xi | \pi(y)\xi \rangle$  is not necessarily an ips-form; the latter fact leads to the notion of regular representation discussed above.

**Example 4.6.** A continuous linear functional  $\omega$  whose restriction to  $\mathfrak{B}$  is positive need not be representable. As an example, consider  $\mathfrak{A} = L^1(I)$ ,  $I$  a bounded interval on the real line, and  $\mathfrak{B} = L^\infty(I)$ . The linear functional

$$\omega(f) = \int_I f(t)dt, \quad f \in L^1(I)$$

is continuous, but it is not representable, since  $(r_3)$  fails if  $f \in L^1(I) \setminus L^2(I)$ .

Since multiplication cores play an important role for topological partial \*-algebras, we restrict our attention to the case where  $\mathfrak{B}$  is a multiplication core and we omit explicit reference to  $\mathfrak{B}$  whenever it appears. We will denote by  $\mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$  the set of  $\tau$ -continuous linear functionals that are representable (with respect to  $\mathfrak{B}$ ).

Since both representable functionals and ips-forms define GNS-like representations it is natural to consider the interplay of these two notions, with particular reference to the topological case. We refer also to [5, 7] for more details.

**Proposition 4.7.** *Let  $\mathfrak{A}$  be a topological partial \*-algebra with multiplication core  $\mathfrak{B}$  which is an algebra. If  $\varphi$  is a  $\tau$ -continuous ips-form on  $\mathfrak{A}$  then, for every  $b \in \mathfrak{B}$ , the linear functional  $\omega_\varphi^b$  defined by*

$$\omega_\varphi^b(x) = \varphi(xb, b), \quad x \in \mathfrak{A}$$

*is representable and the corresponding map  $a \in \mathfrak{B} \mapsto \lambda_{\omega_\varphi^b}^0(a) \in \mathcal{H}_{\omega_\varphi^b}$  is continuous.*

*Conversely, assume that  $\mathfrak{A}$  has a unit  $e \in \mathfrak{B}$ . Then, if  $\omega$  is a representable linear functional on  $\mathfrak{A}$  and the map  $a \in \mathfrak{B} \mapsto \lambda_\omega^0(a) \in \mathcal{H}_\omega$  is continuous, the positive sesquilinear form  $\varphi_\omega$  defined on  $\mathfrak{B} \times \mathfrak{B}$  by*

$$\varphi_\omega(a, b) := \omega(b^*a),$$

*is  $\tau$ -continuous on  $\mathfrak{B} \times \mathfrak{B}$  and it extends to a continuous ips-form  $\tilde{\varphi}_\omega$  on  $\mathfrak{A}$ .*

*Proof.* We prove only the second part of the statement.

For every  $a, b \in \mathfrak{B}$  we have

$$\begin{aligned} |\varphi_\omega(a, b)| &= |\omega(b^*a)| = |\langle \pi_\omega(a)\lambda_\omega^0(e) | \pi_\omega(b)\lambda_\omega^0(e) \rangle| \\ &\leq \|\pi_\omega(a)\lambda_\omega^0(e)\| \|\pi_\omega(b)\lambda_\omega^0(e)\| = \|\lambda_\omega^0(a)\| \|\lambda_\omega^0(b)\| \leq p(a)p(b) \end{aligned}$$

for some continuous seminorm  $p$ .

Hence  $\varphi_\omega$  extends uniquely to  $\mathfrak{A} \times \mathfrak{A}$ . Let  $\tilde{\varphi}_\omega$  denote this extension. It is easily seen that  $\tilde{\varphi}_\omega$  is a positive sesquilinear form on  $\mathfrak{A} \times \mathfrak{A}$  and

$$|\tilde{\varphi}_\omega(x, y)| \leq p(x)p(y), \quad \forall x, y \in \mathfrak{A}.$$

Hence the map  $x \mapsto \lambda_{\tilde{\varphi}_\omega}(x) \in \mathcal{H}_{\tilde{\varphi}_\omega}$  is also continuous, since

$$\|\lambda_{\tilde{\varphi}_\omega}(x)\|^2 = \tilde{\varphi}_\omega(x, x) \leq p(x)^2, \quad \forall x \in \mathfrak{A}.$$

Thus, if  $x = \tau - \lim_\alpha b_\alpha$ ,  $b_\alpha \in \mathfrak{B}$ , we get

$$\|\lambda_{\tilde{\varphi}_\omega}(x) - \lambda_{\tilde{\varphi}_\omega}(b_\alpha)\|^2 = \tilde{\varphi}_\omega(x - b_\alpha, x - b_\alpha) \leq p(x - b_\alpha)^2 \rightarrow 0.$$

The conditions (ips<sub>3</sub>) and (ips<sub>4</sub>) are readily checked. Concerning (ips<sub>4</sub>), for instance, let  $x \in L(y)$  and  $a, b \in \mathfrak{B}$ . Then,

$$\begin{aligned} \tilde{\varphi}_\omega(a, (xy)b) &= \omega(((xy)b)^*a) = \omega(b^*(xy)^*a) \\ &= \langle \pi_\omega(xy)^\dagger \lambda_\omega^0(a) | \lambda_\omega^0(b) \rangle \\ &= \langle (\pi_\omega(y)^\dagger \square \pi_\omega(x)^\dagger) \lambda_\omega^0(a) | \lambda_\omega^0(b) \rangle \\ &= \langle \pi_\omega(x)^\dagger \lambda_\omega^0(a) | \pi_\omega(y) \lambda_\omega^0(b) \rangle \\ &= \tilde{\varphi}_\omega(x^*a, yb). \end{aligned} \quad \square$$

**Remark 4.8.** If  $\omega$  is a representable linear functional on  $\mathfrak{A}$  and the map  $a \in \mathfrak{B} \mapsto \lambda_\omega^0(a) \in \mathcal{H}_\omega$  is continuous, then  $\omega$  is continuous. The converse is false in general.

However, the continuity of  $\omega$  implies the  $\tau^*$ -closability of the map  $\lambda_\omega^0 : a \in \mathfrak{B} \mapsto \lambda_\omega^0(a) \in \mathcal{H}_\omega$  as the next proposition shows.

**Proposition 4.9.** *Let  $\omega$  be continuous and  $\mathfrak{B}$ -positive. Then the map  $\lambda_\omega^0 : a \in \mathfrak{B} \mapsto \lambda_\omega^0(a) \in \mathcal{H}_\omega$  is  $\tau^*$ -closable.*

*Proof.* Let  $a_\delta \xrightarrow{\tau^*} 0$ ,  $a_\delta \in \mathfrak{B}$ , and suppose that the net  $\{\lambda_\omega^0(a_\delta)\}$  is Cauchy in  $\mathcal{H}_\omega$ . Hence it converges to some  $\xi \in \mathcal{H}_\omega$  and

$$\langle \lambda_\omega^0(b) | \lambda_\omega^0(a_\delta) \rangle \rightarrow \langle \lambda_\omega^0(b) | \xi \rangle, \quad \forall b \in \mathfrak{B}.$$

Moreover,

$$\langle \lambda_\omega^0(b) | \lambda_\omega^0(a_\delta) \rangle = \omega(a_\delta^*b) \rightarrow 0, \quad \forall b \in \mathfrak{B},$$

since  $a_\delta^* \xrightarrow{\tau} 0$  and the right multiplication by  $b \in \mathfrak{B}$  and  $\omega$  are both  $\tau$ -continuous. Thus  $\langle \lambda_\omega^0(b) | \xi \rangle = 0$ , for every  $b \in \mathfrak{B}$ . This implies that  $\xi = 0$  and, therefore,  $\lambda_\omega^0(a_\delta) \rightarrow 0$ .  $\square$

Actually, it is easy to see that the closability of the map  $\lambda_\omega^0$  is equivalent to the closability of  $\varphi_\omega$ . Indeed, closability of the map  $a \in \mathfrak{B} \mapsto \lambda_\omega^0(a) \in \mathcal{H}_\omega$  means that if  $a_\delta \xrightarrow{\tau^*} 0$  and  $\{\lambda_\omega^0(a_\delta)\}$  is a Cauchy net, then  $\lambda_\omega^0(a_\delta) \rightarrow 0$ . But  $\{\lambda_\omega^0(a_\delta)\}$  is a Cauchy net if and only if  $\varphi_\omega(a_\delta - a_\gamma, a_\delta - a_\gamma) \rightarrow 0$ . This leads to the conclusion  $\|\lambda_\omega^0(a_\delta)\|^2 = \varphi_\omega(a_\delta, a_\delta) \rightarrow 0$ .

Therefore, Proposition 4.9 generalizes [7, Prop. 2.7], which says that, for a locally convex quasi  $*$ -algebra  $(\mathfrak{A}, \mathfrak{A}_0)$ , the sesquilinear form  $\varphi_\omega$  is closable if  $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ .

Thus, if  $\omega$  is continuous and  $\mathfrak{B}$ -positive, the map  $\lambda_\omega^0$  has a closure  $\overline{\lambda_\omega^0}$  defined on

$$D(\overline{\lambda_\omega^0}) = \{x \in \mathfrak{A} : \exists \{a_\delta\} \subset \mathfrak{B}, a_\delta \xrightarrow{\tau^*} x, \{\lambda_\omega^0(a_\delta)\} \text{ is a Cauchy net}\}.$$

From the discussion above, it follows that  $D(\overline{\lambda_\omega^0})$  coincides with the domain  $D(\overline{\varphi_\omega})$  of the closure of  $\varphi_\omega$ .

For the case of a locally convex quasi \*-algebra  $(\mathfrak{A}, \mathfrak{A}_0)$ , the following assumption was made in [7]:

$$(fr) \quad \bigcap_{\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})} D(\overline{\varphi_\omega}) = \mathfrak{A}.$$

Quasi \*-algebras verifying the condition (fr) are called *fully representable* (hence the acronym). Some concrete examples have been described in [7] and several interesting structure properties have been derived. We maintain the same definition and the same name in the case of topological partial \*-algebras and, in complete analogy, we say that a topological partial \*-algebra  $\mathfrak{A}[\tau]$ , with multiplication core  $\mathfrak{B}$  is *fully representable* if

$$(fr) \quad D(\overline{\varphi_\omega}) = \mathfrak{A}, \text{ for every } \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}).$$

Then we have the following generalization of Proposition 3.6 of [7].

**Proposition 4.10.** *Let  $\mathfrak{A}$  be a semi-associative \*-topological partial \*-algebra with multiplication core  $\mathfrak{B}$ . Assume that  $\mathfrak{A}$  is fully representable and let  $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$ . Then,  $\overline{\varphi_\omega}$  is an ips-form on  $\mathfrak{A}$  with core  $\mathfrak{B}$ , with the property*

$$\overline{\varphi_\omega}(xa, b) = \omega(b^*xa), \quad \forall x \in \mathfrak{A}, a, b \in \mathfrak{B}.$$

*Proof.* The continuity of the involution implies that, for every  $a \in \mathfrak{B}$ , the map  $x \mapsto a^*x$  is continuous on  $\mathfrak{A}$ . Hence the linear functional  $\omega_a$  defined by  $\omega(a^*xa)$  is continuous. We now prove that  $\omega_a$  is representable; for this we need to check properties (r<sub>1</sub>), (r<sub>2</sub>) and (r<sub>3</sub>). We have

$$\omega_a(b^*b) = \omega(a^*(b^*b)a) = \omega((a^*b^*)(ba)) \geq 0, \quad \forall b \in \mathfrak{B},$$

i.e., (r<sub>1</sub>) holds. Furthermore, for every  $b, c \in \mathfrak{B}$ , we have

$$\begin{aligned} \omega_a(c^*(xb)) &= \omega(a^*(c^*(xb))a) = \omega(a^*((c^*x)b)a) \\ &= \overline{\omega(a^*(b^*(x^*c))a)} = \overline{\omega_a(b^*(x^*c))}. \end{aligned}$$

As for (r<sub>3</sub>), for every  $x \in \mathfrak{A}$  and  $b \in \mathfrak{B}$ , we have

$$\begin{aligned} |\omega_a(x^*b)| &= |\omega(a^*(x^*b)a)| = |\omega((a^*x^*)(ba))| \\ &\leq \gamma_{x,a} \omega(a^*(b^*b)a)^{1/2} = \gamma_{x,a} \omega_a(b^*b)^{1/2}. \end{aligned}$$

Thus  $\omega_a \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$ , for every  $a \in \mathfrak{B}$ . By Proposition 4.9,  $\overline{\varphi_{\omega_a}}$  is well-defined and, by the assumption,  $D(\overline{\varphi_{\omega_a}}) = \mathfrak{A}$ . Hence, if  $x \in \mathfrak{A}$ , there exists a net  $\{x_\alpha\} \subset \mathfrak{B}$  such that  $x_\alpha \xrightarrow{\tau^*} x$  and  $\varphi_{\omega_a}(x_\alpha - x_\beta, x_\alpha - x_\beta) \rightarrow 0$  or, equivalently,  $\varphi_\omega((x_\alpha - x_\beta)a, (x_\alpha - x_\beta)a) \rightarrow 0$ . Hence, by the definition of closure, for every  $b \in \mathfrak{A}$ ,

$$\overline{\varphi_\omega}(xa, b) = \lim_{\alpha} \varphi_\omega(x_\alpha a, b) = \lim_{\alpha} \omega(b^*(x_\alpha a)) = \omega(b^*(xa)),$$



by the continuity of  $\omega$ . This easily implies that  $\overline{\varphi_\omega}(xa, b) = \overline{\varphi_\omega}(a, x^*b)$ , for every  $x \in \mathfrak{A}$  and  $a, b \in \mathfrak{B}$ , so that  $\overline{\varphi_\omega}$  satisfies (ips<sub>3</sub>).

Let now  $x \in L(y)$  and  $a, b \in \mathfrak{B}$ . Now let  $\{x_\beta^*\}$  and  $\{y_\alpha\}$  nets in  $\mathfrak{B}$ ,  $\tau^*$ -converging, respectively, to  $x^*$  and  $y$  and such that  $\varphi_\omega((x_\beta^* - x_{\beta'}^*)b, (x_\beta^* - x_{\beta'}^*)b) \rightarrow 0$  and  $\varphi_\omega((y_\alpha - y'_\alpha)a, (y_\alpha - y'_\alpha)a) \rightarrow 0$ . Then we get

$$\begin{aligned} \overline{\varphi_\omega}((xy)a, b) &= \omega(b^*(xy)a) \\ &= \langle \pi_\omega(xy) \lambda_\omega^0(a) | \lambda_\omega^0(b) \rangle \\ &= \langle \pi_\omega(x) \square \pi_\omega(y) \lambda_\omega^0(a) | \lambda_\omega^0(b) \rangle \\ &= \langle \pi_\omega(y) \lambda_\omega^0(a) | \pi_\omega(x^*) \lambda_\omega^0(b) \rangle \\ &= \lim_{\alpha, \beta} \langle \pi_\omega(y_\alpha) \lambda_\omega^0(a) | \pi_\omega(x_\beta^*) \lambda_\omega^0(b) \rangle \\ &= \lim_{\alpha, \beta} \varphi_\omega(y_\alpha a, x_\beta^* b) = \overline{\varphi_\omega}(ya, x^*b). \end{aligned}$$

Thus, (ips<sub>4</sub>) holds. To complete the proof, we need to show that  $\lambda_{\overline{\varphi_\omega}}(\mathfrak{B})$  is dense in the Hilbert space  $\mathcal{H}_{\overline{\varphi_\omega}}$ . This part of the proof is completely analogous to that given in [7, Proposition 3.6] and we omit it.  $\square$

If  $\mathfrak{A}$  is semi-associative and fully representable, every continuous representable linear functional  $\omega$  comes from a closed ips-form  $\overline{\varphi_\omega}$ , but  $\overline{\varphi_\omega}$  need not be continuous, in general, unless more assumptions are made on the topology  $\tau$ .

**Corollary 4.11.** *Let  $\mathfrak{A}[\tau]$  be a fully representable semi-associative  $*$ -topological partial  $*$ -algebra with multiplication core  $\mathfrak{B}$ . Assume that  $\mathfrak{A}[\tau]$  is a Fréchet space. Then, for every  $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$ ,  $\overline{\varphi_\omega}$  is a continuous ips-form.*

*Proof.* The map  $\overline{\lambda_\omega^0}$  is closed and everywhere defined. The closed graph theorem then implies that  $\overline{\lambda_\omega^0}$  is continuous. The statement follows from Proposition 4.7.  $\square$

Summarizing, we have

**Theorem 4.12.** *Let  $\mathfrak{A}[\tau]$  be a fully representable  $*$ -topological partial  $*$ -algebra with multiplication core  $\mathfrak{B}$  and unit  $e \in \mathfrak{B}$ . Assume that  $\mathfrak{A}[\tau]$  is a Fréchet space and the following conditions hold*

- (rc) *Every linear functional  $\omega$  which is continuous and  $\mathfrak{B}$ -positive is representable;*
- (sq) *for every  $x \in \mathfrak{A}$ , there exists a sequence  $\{b_n\} \subset \mathfrak{B}$  such that  $b_n \xrightarrow{\tau} x$  and the sequence  $\{b_n^* b_n\}$  is increasing, in the sense of the order of  $\mathfrak{A}^+(\mathfrak{B})$ .*

*Then  $\mathfrak{A}$  is  $*$ -semisimple.*

*Proof.* Assume, on the contrary, that there exists  $x \in \mathfrak{A} \setminus \{0\}$  such that  $\varphi(x, x) = 0$ , for every  $\varphi \in \mathcal{P}_\mathfrak{B}(\mathfrak{A})$ . If  $\omega$  is continuous and  $\mathfrak{B}$ -positive, then by assumption it is representable; thus  $\overline{\varphi_\omega}$ , which is everywhere defined on  $\mathfrak{A} \times \mathfrak{A}$ , is continuous, by Corollary 4.11. Let  $x = \lim_{n \rightarrow \infty} b_n$ , with  $b_n \in \mathfrak{B}$  and  $\{b_n^* b_n\}$  increasing. Then we have

$$0 \leq \lim_{n \rightarrow \infty} \omega(b_n^* b_n) = \lim_{n \rightarrow \infty} \varphi_\omega(b_n, b_n) = \overline{\varphi_\omega}(x, x) = 0.$$

Then  $\omega(b_n^* b_n) = 0$ , for every  $n \in \mathbb{N}$ . But this contradicts Theorem 4.2.  $\square$

As we have seen in Example 4.6, condition (rc) is not fulfilled in general. To get an example of a situation where this condition is satisfied, it is enough to replace in Example 4.6 the normed partial \*-algebra  $L^1(I)$  with  $L^2(I)$  (which is fully representable, as shown in [7]). It is easily seen that both condition (rc) and (sq) are satisfied in this case. It has been known since a long time that this partial \*-algebra is \*-semisimple [6].

## 5. BOUNDED ELEMENTS IN \*-SEMISIMPLE PARTIAL \*-ALGEBRAS

\*-Semisimple topological partial \*-algebras are characterized by the existence of a sufficient family of ips-forms. This fact was used in [4] and in [7] to derive a number of properties that we want to revisit in this larger framework.

**5.1.  $\mathcal{M}$ -bounded elements.** First we adapt to the present case the definition of  $\mathcal{M}$ -bounded elements given in [4, Def. 4.9] for an  $\mathfrak{A}_0$ -regular topological partial \*-algebra.

**Definition 5.1.** Let  $\mathfrak{A}$  be a topological partial \*-algebra with multiplication core  $\mathfrak{B}$  and a sufficient family  $\mathcal{M}$  of continuous ips-forms with core  $\mathfrak{B}$ . An element  $x \in \mathfrak{A}$  is called  $\mathcal{M}$ -bounded if there exists  $\gamma_x > 0$  such that

$$|\varphi(xa, b)| \leq \gamma_x \varphi(a, a)^{1/2} \varphi(b, b)^{1/2}, \quad \forall \varphi \in \mathcal{M}, a, b \in \mathfrak{B}.$$

An useful characterization of  $\mathcal{M}$ -bounded elements is given by the following proposition, whose proof is similar to that of [4, Proposition 4.10].

**Proposition 5.2.** Let  $\mathfrak{A}[\tau]$  be a topological partial \*-algebra with multiplication core  $\mathfrak{B}$ . Then, an element  $x \in \mathfrak{A}$  is  $\mathcal{M}$ -bounded if, and only if, there exists  $\gamma_x \in \mathbb{R}$  such that  $\varphi(xa, xa) \leq \gamma_x^2 \varphi(a, a)$  for all  $\varphi \in \mathcal{M}$  and  $a \in \mathfrak{B}$ .

If  $x, y$  are  $\mathcal{M}$ -bounded elements and their weak product  $x \square y$  exists, then  $x \square y$  is also  $\mathcal{M}$ -bounded.

**Lemma 5.3.** Let the  $\mathcal{M}$ -bounded element  $x \in \mathfrak{A}$  have a strong inverse  $x^{-1}$ . Then  $\pi(x)$  has a strong inverse for every quasi-symmetric \*-representation  $\pi$ .

*Proof.* Let  $x \in \mathfrak{A}$  with strong inverse  $x^{-1}$ , i.e.,  $x \bullet x^{-1} = x^{-1} \bullet x = e$ . Let  $\pi$  be a \*-representation with  $\pi(e) = I$ , then

$$I = \pi(e) = \pi(x \bullet x^{-1}) = \pi(xx^{-1}) = \pi(x) \square \pi(x^{-1}) = \pi(x) \circ \pi(x^{-1}).$$

It follows that the strong inverse  $\pi(x)^{-1} := \pi(x^{-1})$  of  $\pi(x)$  exists.  $\square$

Given  $X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ , we denote by  $\rho_o^\mathcal{D}(X)$  the set of all complex numbers  $\lambda$  such that  $X - \lambda I_\mathcal{D}$  has a strong bounded inverse [3, Section 3] and by  $\sigma_o^\mathcal{D}(X) := \mathbb{C} \setminus \rho_o^\mathcal{D}(X)$  the corresponding spectrum of  $X$ .

If  $\pi$  is a \*-representation of  $\mathfrak{A}$ , from [3, Proposition 3.9] it follows that  $\sigma_o^\mathcal{D}(\pi(x)) = \sigma(\overline{\pi(x)})$ . If, in particular,  $\pi$  is a quasi-symmetric \*-representation, we can conclude, by Lemma 5.3, that  $\rho^\mathcal{M}(x) \subseteq \rho(\overline{\pi(x)}) = \rho_o^\mathcal{D}(\pi(x))$ , where

$\rho^{\mathcal{M}}(x)$  denotes the set of complex numbers  $\lambda$  such that the strong inverse  $(x - \lambda e)^{-1}$  exists as an  $\mathcal{M}$ -bounded element of  $\mathfrak{A}$  [4, Definition 4.28]. Hence,

$$(6) \quad \sigma(\overline{\pi(x)}) \subseteq \sigma^{\mathcal{M}}(x).$$

Exactly as for partial  $*$ -algebras of operators, there is here a natural distinction between hermitian elements  $x$  of  $\mathfrak{A}$  (i.e.  $x = x^*$ ) and self-adjoint elements (hermitian and with real spectrum).

**Definition 5.4.** *The element  $x \in \mathfrak{A}$  is said  $\mathcal{M}$ -self-adjoint if it is hermitian and  $\sigma^{\mathcal{M}}(x) \subseteq \mathbb{R}$ .*

**Proposition 5.5.** *If  $x \in \mathfrak{A}$  is  $\mathcal{M}$ -self-adjoint, then for every quasi-symmetric  $\pi \in \text{Rep}_c(\mathfrak{A}, \mathfrak{B})$ , the operator  $\pi(x)$  is essentially self-adjoint.*

*Proof.* If  $x \in \mathfrak{A}$  is  $\mathcal{M}$ -self-adjoint, then, for every  $\pi \in \text{Rep}_c(\mathfrak{A}, \mathfrak{B})$ , the operator  $\pi(x)$  is symmetric and  $\sigma^{\mathcal{M}}(x) \subseteq \mathbb{R}$ . By (6) it follows that  $\overline{\pi(x)}$  is self-adjoint, hence  $\pi(x)$  is essentially self-adjoint.  $\square$

## 5.2. Order bounded elements.

**5.2.1. Order structure.** Let  $\mathfrak{A}[\tau]$  be a topological partial  $*$ -algebra with multiplication core  $\mathfrak{B}$ . If  $\mathfrak{A}[\tau]$  is  $*$ -semisimple, there is a natural order on  $\mathfrak{A}$  defined by the family  $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$  or by any sufficient subfamily  $\mathcal{M}$  of  $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ , and this order can be used to define a different notion of *boundedness* of an element  $x \in \mathfrak{A}$  [7, 9, 13].

**Definition 5.6.** Let  $\mathfrak{A}[\tau]$  be a topological partial  $*$ -algebra and  $\mathfrak{B}$  a subspace of  $R\mathfrak{A}$ . A subset  $\mathfrak{K}$  of  $\mathfrak{A}_h := \{x \in \mathfrak{A} : x = x^*\}$  is called a  $\mathfrak{B}$ -admissible wedge if

- (1)  $e \in \mathfrak{K}$ , if  $\mathfrak{A}$  has a unit  $e$ ;
- (2)  $x + y \in \mathfrak{K}$ ,  $\forall x, y \in \mathfrak{K}$ ;
- (3)  $\lambda x \in \mathfrak{K}$ ,  $\forall x \in \mathfrak{K}$ ,  $\lambda \geq 0$ ;
- (4)  $(a^*x)a = a^*(xa) =: a^*xa \in \mathfrak{K}$ ,  $\forall x \in \mathfrak{K}$ ,  $a \in \mathfrak{B}$ .

As usual,  $\mathfrak{K}$  defines an order on the real vector space  $\mathfrak{A}_h$  by  $x \leq y \Leftrightarrow y - x \in \mathfrak{K}$ .

In the rest of this section, we will suppose that the partial  $*$ -algebras under consideration are *semi-associative*. Under this assumption, the first equality in (4) of Definition 5.6 is automatically satisfied.

Let  $\mathfrak{A}$  be a topological partial  $*$ -algebra with multiplication core  $\mathfrak{B}$ . We put

$$\mathfrak{B}^{(2)} = \left\{ \sum_{k=1}^n x_k^* x_k, x_k \in \mathfrak{B}, n \in \mathbb{N} \right\}.$$

If  $\mathfrak{B}$  is a  $*$ -algebra, this is nothing but the set (wedge) of positive elements of  $\mathfrak{B}$ . The  $\mathfrak{B}$ -strongly positive elements of  $\mathfrak{A}$  are then defined as the elements of  $\mathfrak{A}^+(\mathfrak{B}) := \overline{\mathfrak{B}^{(2)}}^{\tau}$ , already defined in Section 4. Since  $\mathfrak{A}$  is semi-associative, the set  $\mathfrak{A}^+(\mathfrak{B})$  of  $\mathfrak{B}$ -strongly positive elements is a  $\mathfrak{B}$ -admissible wedge.

We also define

$$\mathfrak{A}_{\text{alg}}^+ = \left\{ \sum_{k=1}^n x_k^* x_k, x_k \in R\mathfrak{A}, n \in \mathbb{N} \right\},$$

the set (wedge) of positive elements of  $\mathfrak{A}$  and we put  $\mathfrak{A}_{\text{top}}^+ := \overline{\mathfrak{A}_{\text{alg}}^+}^\tau$ . The semi-associativity implies that  $R\mathfrak{A} \cdot R\mathfrak{A} \subseteq R\mathfrak{A}$  and then  $\mathfrak{A}_{\text{top}}^+$  is  $R\mathfrak{A}$ -admissible.

Let  $\mathcal{M} \subseteq \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ . An element  $x \in \mathfrak{A}$  is called  $\mathcal{M}$ -positive if

$$\varphi(xa, a) \geq 0, \quad \forall \varphi \in \mathcal{M}, a \in \mathfrak{B}.$$

An  $\mathcal{M}$ -positive element is automatically hermitian. Indeed, if  $\varphi(xa, a) \geq 0$ ,  $\forall \varphi \in \mathcal{M}, \forall a \in \mathfrak{B}$ , then  $\varphi(a, x^*a) = \varphi(xa, a) \geq 0$  and  $\varphi(x^*a, a) \geq 0$ ; hence  $\varphi((x - x^*)a, a) = 0, \forall \varphi \in \mathcal{M}, \forall a \in \mathfrak{B}$ . By (iii) of Lemma 3.7, it follows that  $x = x^*$ .

We denote by  $\mathfrak{A}_{\mathcal{M}}^+$  the set of all  $\mathcal{M}$ -positive elements. Clearly  $\mathfrak{A}_{\mathcal{M}}^+$  is a  $\mathfrak{B}$ -admissible wedge.

**Proposition 5.7.** *The following inclusions hold*

$$(7) \quad \mathfrak{A}^+(\mathfrak{B}) \subseteq \mathfrak{A}_{\text{top}}^+ \subseteq \mathfrak{A}_{\mathcal{M}}^+, \quad \forall \mathcal{M} \subseteq \mathcal{P}_{\mathfrak{B}}(\mathfrak{A}).$$

*Proof.* We only prove the second inclusion. Let  $x \in \mathfrak{A}_{\text{top}}^+$ . Then  $x = \lim_{\alpha} b_{\alpha}$  with  $b_{\alpha} = \sum_{i=1}^n c_{\alpha,i}^* c_{\alpha,i}, c_{\alpha,i} \in R\mathfrak{A}$ . Thus,

$$\begin{aligned} \varphi(xa, a) &= \lim_{\alpha} \varphi(b_{\alpha}a, a) = \lim_{\alpha} \varphi\left(\sum_i (c_{\alpha,i}^* c_{\alpha,i})a, a\right) \\ &= \lim_{\alpha} \sum_i \varphi((c_{\alpha,i}^* c_{\alpha,i})a, a) = \lim_{\alpha} \sum_i \varphi(c_{\alpha,i}a, c_{\alpha,i}a) \geq 0. \end{aligned}$$

by (ips<sub>4</sub>). □

Of course, one expects that under certain conditions the converse inclusions hold, or that the three sets in (7) actually coincide. A partial answer is given in Corollary 5.16.

**Example 5.8.** We give here two examples where the wedges considered above coincide.

(1) The first example, very elementary, is obtained by considering the space  $L^p(X)$ ,  $p \geq 2$ . Indeed, it is easily seen that the  $\mathcal{M}$ -positivity of a function  $f$  simply means that  $f(t) \geq 0$  a.e. in  $X$  ( $\mathcal{M}$  is here the family of ips-forms defined in Example 3.9). On the other hand it is well-known that such a function can be approximated in norm by a sequence of nonnegative functions of  $L^{\infty}(X)$ .

(2) Let  $T$  be a self-adjoint operator with dense domain  $D(T)$  and denote by  $E(\cdot)$  the spectral measure of  $T$ . We consider the space  $\mathcal{L}^{\dagger}(\mathcal{D}, \mathcal{H})$  where  $D := \mathcal{D}^{\infty}(T) = \bigcap_{n \in \mathbb{N}} D(T^n)$ . We prove that if  $\mathcal{L}^{\dagger}(\mathcal{D}, \mathcal{H})$  is endowed with the topology  $\mathfrak{t}_{s*}$  and  $\mathcal{M}$  is the family of ips-forms defined in Example 3.8, then every  $X \in \mathcal{L}^{\dagger}(\mathcal{D}, \mathcal{H})$  which is  $\mathcal{M}$ -positive, i.e.,  $\langle X\xi | \xi \rangle \geq 0$ , for every  $\xi \in \mathcal{D}$ , is the  $\mathfrak{t}_{s*}$ -limit of elements of  $\mathbb{L}_b^{\dagger}(\mathcal{D})^{(2)}$ . Indeed, if  $\Delta, \Delta'$  are bounded Borel subsets of the real line, then  $E(\Delta^{(j)})\xi \in \mathcal{D}$ , for every  $\xi \in \mathcal{H}$ . This implies

that  $E(\Delta')YE(\Delta)$  is a bounded operator in  $\mathcal{H}$ , for every  $Y \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  and its restriction to  $\mathcal{D}$  belongs to  $\mathbb{L}_b^\dagger(\mathcal{D})$ . Put  $\Delta_N = (-N, N]$ ,  $N \in \mathbb{N}$  and  $\delta_m = (m, m+1]$ ,  $m \in \mathbb{Z}$ . The  $\mathcal{M}$ -positivity of  $X$  implies that  $E(\Delta_N)XE(\Delta_N) = B_N^*B_N$  for some bounded operator  $B_N$ . Then, observing that  $\sum_{m \in \mathbb{Z}} E(\delta_m) = I$ , in strong or strong\*-sense, we obtain

$$\begin{aligned} E(\Delta_N)XE(\Delta_N) &= B_N^*B_N = E(\Delta_N)B_N^*B_NE(\Delta_N) \\ &= E(\Delta_N)B_N^* \left( \sum_{m \in \mathbb{Z}} E(\delta_m) \right) B_NE(\Delta_N) \\ &= \sum_{m \in \mathbb{Z}} (E(\Delta_N)B_N^*E(\delta_m))(E(\delta_m)B_NE(\Delta_N)). \end{aligned}$$

This proves that  $E(\Delta_N)XE(\Delta_N)$  belongs to the  $\mathfrak{t}_{s^*}$ -closure of  $\mathbb{L}_b^\dagger(\mathcal{D})$ <sup>(2)</sup> Now, if we let  $N \rightarrow \infty$ , we easily get  $\|X\xi - E(\Delta_N)XE(\Delta_N)\xi\| \rightarrow 0$  and so the statement is proved.

An improvement of Theorem 4.2 is provided by the following

**Corollary 5.9.** *Let  $\mathcal{M}$  be sufficient. Then, for every  $x \in \mathfrak{A}_{\mathcal{M}}^+$ ,  $x \neq 0$ , there exists  $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$  with the properties*

- (a)  $\omega(y) \geq 0$ ,  $\forall y \in \mathfrak{A}_{\mathcal{M}}^+$ ;
- (b)  $\omega(x) > 0$ .

*Proof.* By the previous proposition, if  $x \in \mathfrak{A}_{\mathcal{M}}^+$ ,  $x \neq 0$ , there exist  $\varphi \in \mathcal{M}$  and  $a \in \mathfrak{B}$  such that  $\varphi(xa, a) > 0$ . Hence the linear functional  $\omega(y) := \varphi(ya, a)$  has the desired properties.  $\square$

**Proposition 5.10.** *Let the family  $\mathcal{M}$  be sufficient. Then,  $\mathfrak{A}_{\mathcal{M}}^+$  is a cone, i.e.,  $\mathfrak{A}_{\mathcal{M}}^+ \cap (-\mathfrak{A}_{\mathcal{M}}^+) = \{0\}$ .*

*Proof.* If  $x \in \mathfrak{A}_{\mathcal{M}}^+ \cap (-\mathfrak{A}_{\mathcal{M}}^+)$ , then  $\varphi(xa, a) \geq 0$  and  $\varphi((-x)a, a) \geq 0$ , for every  $\varphi \in \mathcal{M}$  and  $a \in \mathfrak{B}$ . Hence  $\varphi(xa, a) = 0$ , for every  $\varphi \in \mathcal{M}$  and  $a \in \mathfrak{B}$ . The sufficiency of  $\mathcal{M}$  then implies  $x = 0$ .  $\square$

**Remark 5.11.** The fact that  $\mathfrak{A}_{\mathcal{M}}^+$  is a cone automatically implies that  $\mathfrak{A}^+(\mathfrak{B})$  is a cone too.

The following statement shows that  $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ -positivity is exactly what is needed if we want the order to be preserved under any continuous \*-representation. A partially equivalent statement is given in [7, Proposition 3.1]. For making the notations lighter, we put  $\mathfrak{A}_{\mathcal{P}}^+ := \mathfrak{A}_{\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})}^+$ .

**Proposition 5.12.** *Let  $\mathfrak{A}$  be a topological partial \*-algebra with multiplication core  $\mathfrak{B}$  and unit  $e \in \mathfrak{B}$ . Then, the element  $x \in \mathfrak{A}$  belongs to  $\mathfrak{A}_{\mathcal{P}}^+$  if and only if the operator  $\pi(x)$  is positive for every  $(\tau, \mathfrak{t}_s)$ -continuous \*-representation  $\pi$  with  $\pi(e) = I_{\mathcal{D}(\pi)}$ .*

*Proof.* Let  $x \in \mathfrak{A}_{\mathcal{P}}^+$  and let  $\pi$  be a  $(\tau, \mathbf{t}_s)$ -continuous \*-representation of  $\mathfrak{A}$  with  $\pi(e) = I_{\mathcal{D}(\pi)}$ . The sesquilinear form  $\varphi_{\pi}^{\xi}$ , defined by

$$\varphi_{\pi}^{\xi}(x, y) := \langle \pi(x)\xi | \pi(y)\xi \rangle, \quad x, y \in \mathfrak{A},$$

is a continuous ips-form as shown in the proof of Proposition 3.2. Then,

$$\varphi_{\pi}^{\xi}(xa, a) = \langle \pi(xa)\xi | \pi(a)\xi \rangle = \langle (\pi(x) \square \pi(a))\xi | \pi(a)\xi \rangle;$$

in particular, for  $a = e$ ,  $\langle \pi(x)\xi | \xi \rangle \geq 0$ .

Conversely, let  $\varphi \in \mathcal{P}$  and  $\pi_{\varphi}$  the corresponding GNS representation. Then, as remarked in the proof of Proposition 3.2,  $\pi_{\varphi}$  is  $(\tau, \mathbf{t}_s)$ -continuous. We have, for every  $a \in \mathfrak{B}$ ,

$$\varphi(xa, a) = \langle \pi_{\varphi}(x)\lambda_{\varphi}(a) | \lambda_{\varphi}(a) \rangle \geq 0,$$

i.e.,  $x \in \mathfrak{A}_{\mathcal{P}}^+$ . □

**Proposition 5.13.** *Let  $\mathfrak{A}$  be a fully-representable \*-topological partial \*-algebra with multiplication core  $\mathfrak{B}$  and unit  $e \in \mathfrak{B}$ . Assume that  $\mathfrak{A}[\tau]$  is a Fréchet space. Then the following statements are equivalent:*

- (i)  $x \in \mathfrak{A}_{\mathcal{P}}^+$ ;
- (ii)  $\omega(x) \geq 0, \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$ .

*Proof.* (i) $\Rightarrow$ (ii): If  $x \in \mathfrak{A}_{\mathcal{P}}^+$ , then  $\varphi(xa, a) \geq 0, \forall \varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A}), \forall a \in \mathfrak{A}_0$ . If  $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$ , by the assumptions and by Proposition 4.10 it follows that  $\overline{\varphi_{\omega}}$  is an everywhere defined ips-form and thus, by Corollary 4.11, it is continuous. Hence,

$$\omega(a^*xa) = \overline{\varphi_{\omega}}(xa, a) \geq 0, \quad \forall a \in \mathfrak{B}.$$

For  $a = e$ , we get that  $\omega(x) \geq 0$ .

(ii) $\Rightarrow$ (i): If  $\omega(x) \geq 0, \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$ , then this also holds for every linear functional  $\omega_{\varphi}^a, a \in \mathfrak{B}$ , defined by  $\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$  as in Proposition 4.7. Then

$$\varphi(xa, a) = \omega_{\varphi}^a(x) \geq 0, \quad \forall \varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A}).$$

By definition, this means that  $x \in \mathfrak{A}_{\mathcal{P}}^+$ . □

In complete analogy with Proposition 3.9 of [7], one can prove the following

**Proposition 5.14.** *Let  $\mathfrak{A}[\tau]$  be a \*-semisimple \*-topological partial \*-algebra with multiplication core  $\mathfrak{B}$ .*

*Assume that the following condition (P) holds:*

- (P)  $y \in \mathfrak{A}$  and  $\omega(a^*ya) \geq 0$ , for every  $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$  and  $a \in \mathfrak{A}_0$ ,  
imply  $y \in \mathfrak{A}^+(\mathfrak{B})$ .

*Then, for an element  $x \in \mathfrak{A}$ , the following statements are equivalent:*

- (i)  $x \in \mathfrak{A}^+(\mathfrak{B})$ ;
- (ii)  $\omega(x) \geq 0$ , for every  $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$ ;
- (iii)  $\pi(x) \geq 0$ , for every  $(\tau, \mathbf{t}_w)$ -continuous \*-representation  $\pi$  of  $\mathfrak{A}$ .

**Remark 5.15.** In [7, Proposition 3.9] it was required that the family  $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$  of continuous linear functionals does not annihilate positive elements. This is always true for  $*$ -semisimple partial  $*$ -algebras, because of Proposition 5.10.

The previous propositions allow to compare the different cones defined so far.

**Corollary 5.16.** *Under the assumptions of Propositions 5.13 and 5.14, one has  $\mathfrak{A}^+(\mathfrak{B}) = \mathfrak{A}_p^+$ .*

**5.2.2. Order bounded elements.** Let  $\mathfrak{A}[\tau]$  be a topological partial  $*$ -algebra with multiplication core  $\mathfrak{B}$  and unit  $e \in \mathfrak{B}$ . As we have seen in Section 5.2.1,  $\mathfrak{A}[\tau]$  has several natural orders, all related to the topology  $\tau$ . Each one of them can be used to define *bounded* elements. We begin in a purely algebraic way starting from an arbitrary  $\mathfrak{B}$ -admissible cone  $\mathfrak{K}$ .

Let  $x \in \mathfrak{A}$ ; put  $\Re(x) = \frac{1}{2}(x + x^*)$ ,  $\Im(x) = \frac{1}{2i}(x - x^*)$ . Then  $\Re x, \Im(x) \in \mathfrak{A}_h$  (the set of self-adjoint elements of  $\mathfrak{A}$ ) and  $x = \Re(x) + i\Im(x)$ .

**Definition 5.17.** An element  $x \in \mathfrak{A}$  is called  *$\mathfrak{K}$ -bounded* if there exists  $\gamma \geq 0$  such that

$$\pm \Re(x) \leq \gamma e; \quad \pm \Im(x) \leq \gamma e.$$

We denote by  $\mathfrak{A}_b(\mathfrak{K})$  the family of  $\mathfrak{K}$ -bounded elements.

The following statements are easily checked.

- (1)  $\alpha x + \beta y \in \mathfrak{A}_b(\mathfrak{K}), \quad \forall x, y \in \mathfrak{A}_b(\mathfrak{K}), \alpha, \beta \in \mathbb{C}$ .
- (2)  $x \in \mathfrak{A}_b(\mathfrak{K}) \Leftrightarrow x^* \in \mathfrak{A}_b(\mathfrak{K})$ .

**Remark 5.18.** If  $\mathfrak{A}$  is a  $*$ -algebra then, as shown in [9, Lemma 2.1], one also has

- (3)  $x, y \in \mathfrak{A}_b(\mathfrak{K}) \Rightarrow xy \in \mathfrak{A}_b(\mathfrak{K})$ .
- (4)  $a \in \mathfrak{A}_b(\mathfrak{K}) \Leftrightarrow aa^* \in \mathfrak{A}_b(\mathfrak{K})$ .

These statements do not hold in general when  $\mathfrak{A}$  is a partial  $*$ -algebra. They are true, of course, for elements of  $\mathfrak{B}$ .

For  $x \in \mathfrak{A}_h$ , put

$$\|x\|_b := \inf\{\gamma > 0 : -\gamma e \leq x \leq \gamma e\}.$$

$\|\cdot\|_b$  is a seminorm on the real vector space  $(\mathfrak{A}_b(\mathfrak{K}))_h$ .

**Lemma 5.19.** *Let  $\mathcal{M}$  be sufficient. If  $\mathfrak{K} = \mathfrak{A}_{\mathcal{M}}^+$ , then  $\|\cdot\|_b$  is a norm on  $(\mathfrak{A}_b(\mathcal{M}))_h$ .*

*Proof.* By Proposition 5.10,  $\mathfrak{A}_{\mathcal{M}}^+$  is a cone. Put  $E = \{\gamma > 0 : -\gamma e \leq x \leq \gamma e\}$ . If  $\inf E = 0$ , then, for every  $\epsilon > 0$ , there exists  $\gamma_\epsilon \in E$  such that  $\gamma_\epsilon < \epsilon$ . This implies that  $-\epsilon e \leq x \leq \epsilon e$ . If  $\varphi \in \mathcal{M}$ , we get  $-\epsilon \varphi(a, a) \leq \varphi(xa, a) \leq \epsilon \varphi(a, a)$ , for every  $a \in \mathfrak{B}$ . Hence,  $\varphi(xa, a) = 0$ . By the sufficiency of  $\mathcal{M}$ , it follows that  $x = 0$ .  $\square$

Let  $\mathfrak{A}[\tau]$  be a  $*$ -semisimple topological partial  $*$ -algebra with multiplication core  $\mathfrak{B}$ . We can then specify the wedge  $\mathfrak{K}$  as one of those defined above.



Take first  $\mathfrak{K} = \mathfrak{A}_{\mathcal{M}}^+$ , where  $\mathcal{M} = \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$  is the sufficient family of all continuous ips-forms with core  $\mathfrak{B}$ . For simplicity, we write again  $\mathcal{P} := \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ , hence  $\mathfrak{A}_{\mathcal{P}}^+ := \mathfrak{A}_{\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})}^+$  and  $\mathfrak{A}_b(\mathcal{P}) := \mathfrak{A}_b(\mathcal{P}_{\mathfrak{B}}(\mathfrak{A}))$ .

**Proposition 5.20.** *If  $x \in \mathfrak{A}_b(\mathcal{P})$ , then  $\pi(x)$  is a bounded operator, for every  $(\tau, \mathbf{t}_s)$ -continuous \*-representation of  $\mathfrak{A}$ . Moreover, if  $x = x^*$ ,  $\|\pi(x)\| \leq \|x\|_b$ .*

*Proof.* This follows easily from Proposition 5.12 and from the definitions.  $\square$

The following theorem generalizes [7, Theorem 5.5 ].

**Theorem 5.21.** *Let  $\mathfrak{A}[\tau]$  be a fully representable, semi-associative \*-topological partial \*-algebra, with multiplication core  $\mathfrak{B}$  and unit  $e \in \mathfrak{B}$ . Assume that  $\mathfrak{A}[\tau]$  is a Fréchet space. Then the following statements are equivalent:*

(i)  $x \in \mathfrak{A}_b(\mathcal{P})$ .

(ii) There exists  $\gamma_x > 0$  such that

$$|\omega(a^*xa)| \leq \gamma_x \omega(a^*a), \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}), \forall a \in \mathfrak{B}.$$

(iii) There exists  $\gamma_x > 0$  such that

$$|\omega(b^*xa)| \leq \gamma_x \omega(a^*a)^{1/2} \omega(b^*b)^{1/2}, \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}), \forall a, b \in \mathfrak{B}.$$

*Proof.* It is sufficient to consider the case  $x = x^*$ .

(i)  $\Rightarrow$  (iii) If  $x = x^* \in \mathfrak{A}_b(\mathcal{P})$ , there exists  $\gamma > 0$  such that  $-\gamma e \leq x \leq \gamma e$ ; or, equivalently,

$$-\gamma \varphi(a, a) \leq \varphi(xa, a) \leq \gamma \varphi(a, a), \forall \varphi \in \mathcal{P}, a \in \mathfrak{B}.$$

Since  $\mathfrak{A}$  is fully representable,  $D(\overline{\varphi_\omega}) = \mathfrak{A}$  and, by Corollary 4.11, it is a continuous ips-form with core  $\mathfrak{B}$ . Thus, as seen in the proof of Proposition 3.2,  $\pi_{\overline{\varphi_\omega}}$  is  $(\tau, \mathbf{t}_s)$ -continuous. Hence, by Proposition 5.20,  $\pi_{\overline{\varphi_\omega}}(x)$  is bounded and  $\|\pi_{\overline{\varphi_\omega}}(x)\| \leq \|x\|_b$ . Therefore,

$$\begin{aligned} |\omega(b^*xa)| &= |\overline{\varphi_\omega(xa, b)}| \leq \overline{\varphi_\omega(xa, xa)}^{1/2} \varphi_\omega(b, b)^{1/2} \\ &= \|\pi_{\overline{\varphi_\omega}}(x) \lambda_{\overline{\varphi_\omega}}(a)\| \varphi_\omega(b, b)^{1/2} \leq \|x\|_b \gamma_x \omega(a^*a)^{1/2} \omega(b^*b)^{1/2}. \end{aligned}$$

(iii)  $\Rightarrow$  (ii) is obvious.

(ii)  $\Rightarrow$  (i) Assume now that there exists  $\gamma_x > 0$  such that

$$(8) \quad |\omega(a^*xa)| \leq \gamma_x \omega(a^*a), \quad \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}), a \in \mathfrak{B}.$$

Define

$$\tilde{\gamma} := \sup\{|\omega(a^*xa)| : \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0), a \in \mathfrak{A}_0, \omega(a^*a) = 1\}.$$

Let  $\varphi \in \mathcal{P}$  and  $a \in \mathfrak{B}$ . By Proposition 4.7, the linear functional  $\omega_\varphi^a$  defined by  $\omega_\varphi^a(x) = \varphi(xa, a)$ ,  $x \in \mathfrak{A}$ , is continuous and representable. If  $\varphi(a, a) = 0$ , then, by (8),  $\varphi(xa, a) = 0$ . If  $\varphi(a, a) > 0$ , we get

$$\varphi((\tilde{\gamma}e \pm x)a, a) = \tilde{\gamma} \varphi(a, a) \pm \varphi(xa, a) = \varphi(a, a)(\tilde{\gamma} \pm \varphi(xu, u)) \geq 0,$$

where  $u = a\varphi(a, a)^{-1/2}$ . Hence, by the arbitrariness of  $\varphi$  and  $a$ , we have  $x \in \mathfrak{A}_b(\mathcal{P})$ .  $\square$

We can now compare the notion of order bounded element with that of  $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ -bounded element given in Definition 5.1.

**Theorem 5.22.** *Let  $\mathfrak{A}[\tau]$  be a  $*$ -semisimple topological partial  $*$ -algebra with multiplication core  $\mathfrak{B}$  and unit  $e \in \mathfrak{B}$ . For  $x \in \mathfrak{A}$ , the following statements are equivalent.*

- (i)  $x$  is  $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ -bounded.
- (ii)  $x \in \mathfrak{A}_b(\mathcal{P})$ .
- (iii)  $\pi(x)$  is bounded, for every  $\pi \in \text{Rep}_c(\mathfrak{A})$ , and

$$\sup\{\|\overline{\pi(x)}\|, \pi \in \text{Rep}_c(\mathfrak{A})\} < \infty.$$

*Proof.* It is sufficient to consider the case  $x = x^*$ .

(i)  $\Rightarrow$  (ii): If  $x = x^*$  is  $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ -bounded, we have, for some  $\gamma > 0$ ,

$$-\gamma\varphi(a, a) \leq \varphi(xa, a) \leq \gamma\varphi(a, a), \quad \forall \varphi \in \mathcal{P}, a \in \mathfrak{B}.$$

This means that  $-\gamma e \leq x \leq \gamma e$  in the sense of the order induced by  $\mathfrak{A}_{\mathcal{P}}^+$ . Hence  $x \in \mathfrak{A}_b(\mathcal{P})$ .

(ii)  $\Rightarrow$  (iii): Let  $\pi \in \text{Rep}_c(\mathfrak{A})$  and  $\xi \in \mathcal{D}(\pi)$ . Define  $\varphi_{\pi}^{\xi}$  as in the proof of Proposition 5.12. Then  $\varphi_{\pi}^{\xi} \in \mathcal{P}$ . Hence by (ii),  $|\varphi_{\pi}^{\xi}(xa, a)| \leq \gamma_x \varphi_{\pi}^{\xi}(a, a)$ , for some  $\gamma_x > 0$  which depends on  $x$  only. In other words,  $|\langle \pi(x)\xi | \xi \rangle| \leq \gamma_x \|\xi\|^2$ . This in turn easily implies that  $|\langle \pi(x)\xi | \eta \rangle| \leq \gamma_x \|\xi\| \|\eta\|$ , for every  $\xi, \eta \in \mathcal{D}(\pi)$ . Hence  $\pi(x)$  is bounded and  $\|\pi(x)\| \leq \gamma_x$ .

(iii)  $\Rightarrow$  (i): Put  $\gamma_x := \sup\{\|\overline{\pi(x)}\|, \pi \in \text{Rep}_c(\mathfrak{A})\}$ . Then

$$|\langle \pi(x)\xi | \xi \rangle| \leq \|\pi(x)\xi\| \|\xi\| \leq \gamma_x \|\xi\|^2, \quad \forall \xi \in \mathcal{D}_{\pi}.$$

This in particular holds for the GNS representation  $\pi_{\varphi}$  associated to any  $\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ , since  $\pi_{\varphi}$  is  $(\tau, \mathbf{t}_s)$ -continuous. Hence, for every  $a \in \mathfrak{B}$ , we get

$$|\varphi(xa, a)| = |\langle \pi_{\varphi}(x)\lambda_{\varphi}(a) | \lambda_{\varphi}(a) \rangle| \leq \gamma_x \|\lambda_{\varphi}(a)\|^2 = \gamma_x \varphi(a, a).$$

Using the polarization identity, one finally gets

$$|\varphi(xa, b)| \leq \gamma_x \varphi(a, a)^{1/2} \varphi(b, b)^{1/2}, \quad \forall \varphi \in \mathcal{P}, a, b \in \mathfrak{B}.$$

This proves that  $x$  is  $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ -bounded.  $\square$

Theorem 5.22 shows that, under the assumptions we have made, order boundedness is nothing but the  $\mathcal{M}$ -boundedness studied in [4]. So all results proved there apply to the present situation (in particular those concerning the structure of the topological partial  $*$ -algebra under consideration and its spectral properties). Clearly, the crucial assumption is the existence of sufficiently many continuous ips-forms, that is, the  $*$ -semisimplicity.

**Example 5.23.** In particular, Theorem 5.22 shows that, in  $\mathcal{L}^{\dagger}(\mathcal{D}, \mathcal{H})$  (see Example 3.8 for notations), bounded elements defined by  $\mathcal{M}$  and those defined by the order coincide and (as expected) the family of bounded elements is  $L_b^{\dagger}(\mathcal{D}, \mathcal{H})$ . Of course, one could get this result directly, using well-known properties of operators.

Also in the case of  $L^p$ -spaces ( $p > 2$ ) considered in Example 3.9, one obtains that the two notions of boundedness coincide and the bounded part is exactly  $L^{\infty}(X)$ , as can also be proved by elementary arguments.

So far we have considered the order boundedness defined by the cone  $\mathfrak{A}_P^+$ , but other choices are possible. For instance we may consider the order induced by  $\mathfrak{A}^+(\mathfrak{B})$ . It is clear that if  $x \in \mathfrak{A}_b(\mathfrak{A}^+(\mathfrak{B}))$  then  $x \in \mathfrak{A}_b(\mathcal{P})$ . On the other hand, if  $x \in \mathfrak{A}_b(\mathcal{P})$  and the assumptions of Theorem 5.21 hold, there exists  $\gamma_x > 0$  such that

$$|\omega(a^*xa)| \leq \gamma_x \omega(a^*a), \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}), \forall a \in \mathfrak{B}.$$

Hence, if condition (P) holds too, we can conclude, by adapting the argument used in the proof of Theorem 5.21, that  $x \in \mathfrak{A}_b(\mathfrak{A}^+(\mathfrak{B}))$ . We leave a deeper analysis of the general question to future papers.

## 6. CONCLUDING REMARKS

As we have discussed in the Introduction, the notion of bounded element for a topological partial \*-algebra plays an important role for the whole discussion. We have at hand two different notions, one ( $\mathcal{M}$ -boundedness) based on a sufficient family of ips-forms, and another one (order boundedness) based on some  $\mathfrak{B}$ -admissible wedge, where  $\mathfrak{B}$  is a multiplication core. Both seem very reasonable definitions and, as we have seen, they can be compared in many occasions. In the framework of (topological) \*-algebras, it is even possible that every element is order bounded (see examples in [10, Section 5]).<sup>3</sup> The analogous situation for partial \*-algebras is unsolved (in other words we do not know if there exist topological partial \*-algebras where every element is order bounded) and we conjecture that a *complete* topological partial \*-algebra  $\mathfrak{A}$  whose elements are all bounded is necessarily an algebra. This is certainly true in the case where  $\mathcal{M}$ -boundedness is considered, where  $\mathcal{M}$  is a well-behaved family of ips-forms in the sense of Definition 4.26 of [4]. Indeed, as shown there (Proposition 4.27), under these assumptions the set of  $\mathcal{M}$ -bounded elements is a C\*-algebra. The same, of course, holds true in the situation considered in Theorem 5.22, if the family  $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$  is well-behaved. However, the general question is open.

## REFERENCES

- [1] J-P. Antoine, A. Inoue and C. Trapani, *Partial \*-algebras and their operator realizations*, Kluwer, Dordrecht, 2002.
- [2] J-P. Antoine, C. Trapani and F. Tschinke, *Continuous \*-homomorphisms of Banach Partial \*-algebras*, Mediterr. j. math. 4 (2007), 357–373.
- [3] J-P. Antoine, C. Trapani and F. Tschinke, *Spectral properties of partial \*-algebras* Mediterr. j. math. 7 (2010), 123–142.
- [4] J-P. Antoine, C. Trapani and F. Tschinke, *Bounded elements in certain topological partial \*-algebras*, Studia Math. 203 (2011), 223–251.
- [5] F. Bagarello, A. Inoue and C. Trapani, *Representable linear functionals on partial \*-algebras*, Mediterr. j. math. 9 (2012), 153–163.
- [6] F. Bagarello and C. Trapani,  *$L^p$ -spaces as quasi \*-algebras*, J. Math. Anal. Appl. 197 (1996), 810–824.
- [7] M. Fragoulopoulou, C. Trapani and S. Triolo, *Locally convex quasi \*-algebras with sufficiently many \*-representations*, J. Math. Anal. Appl. 388 (2012), 1180–1193.

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<sup>3</sup> The terminology adopted in that paper comes from algebraic geometry, so that an admissible cone is called there a *quadratic module*.

- [8] K. Schmüdgen, *Unbounded operator algebras and representation theory*, Birkhäuser Verlag, Basel, 1990.
- [9] K. Schmüdgen, *A strict Positivstellensatz for the Weyl algebra*, Math. Ann. 331 (2005), 779–794.
- [10] K. Schmüdgen, *Noncommutative real algebraic geometry - Some basic concepts and first ideas*, in *Emerging Applications in Algebraic Geometry*, ed. by M. Putinar and S. Sullivant, Springer, 2009.
- [11] C. Trapani, *\*-Representations, seminorms and structure properties of normed \*-algebras*, Studia Mathematica, Vol. 186, 47-75 (2008).
- [12] C. Trapani and F. Tschinke, *Unbounded  $C^*$ -seminorms and biweights on partial \*-algebras* Mediterr. j. math. 2 (2005) 301–313.
- [13] I. Vidav, *On some \*-regular rings*, Acad. Serbe Sci. Publ. Inst. Math. 13 (1959) 73–80.

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